Pregroup representable expansions of residuated lattices

Andrew Craig   Claudette Robinson

UNIVERSITY OF JOHANNESBURG

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Relation algebras

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and

1. $x^{\sqcup\sqcup} = x$
2. $(xy)^{\sqcup} = y^{\sqcup} x^{\sqcup}$
3. $x(y \vee z) = xy \vee xz$
4. $(x \vee y)^{\sqcup} = x^{\sqcup} \vee y^{\sqcup}$
5. $x^{\sqcup}(xy)' \leqslant y'$

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Let X be a set and $\mathrm{id}_X = \{(x, x) \mid x \in X\}$. Then

$$\mathcal{A}(X) = \langle \wp(X^2), \cap, \cup, ^{\mathsf{c}}, \emptyset, X^2, ;, \mathrm{id}_X, ^{\smile} \rangle$$

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Relation algebra $\mathbf{A}$ is representable if $\mathbf{A} \in \mathbb{ISP}(\{\mathcal{A}(X) \mid X \text{ a set}\})$

Group representable RAs

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Theorem (McKinsey 1940s)

If $\mathbf{G}$ is a group then $\mathcal{R}(\mathbf{G}) = \langle \wp(G), \cap, \cup, ^{\mathsf{c}}, \emptyset, G, \bullet, \{e\}, ^{-1} \rangle$ is a relation algebra.

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*If $\mathbf{A} \in$ **RA** and $\mathbf{A} \approx \mathcal{R}(\mathbf{G})$ for some group $\mathbf{G}$, then $\mathbf{A} \in$ **RRA**.*

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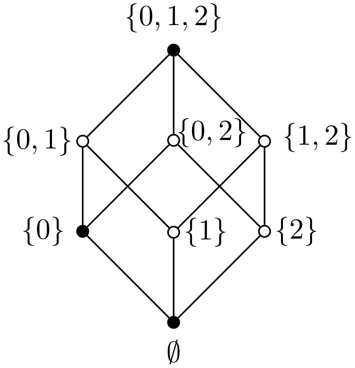
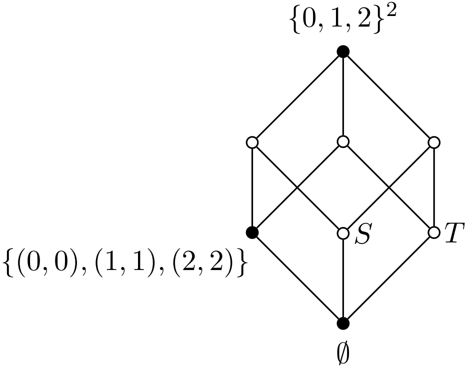
Proof.

The map $\sigma : \wp(G) \to \wp(G \times G)$ defined by $\sigma(Y) = \{ (g, g \cdot y) \mid g \in G, y \in Y \}$ is an embedding of $\mathcal{R}(\mathbf{G})$ into $\mathcal{A}(G)$. □

Example: group representable relation algebra



$\mathcal{R}(\mathbb{Z}_3)$



$S = \{(0,1),(1,2),(2,0)\}$
$T = \{(0,2),(1,0),(2,1)\}$

$\sigma(\mathcal{R}(\mathbb{Z}_3))$

Generalisations of relation algebras

$\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ is a residuated lattice (RL) if $\langle A, \wedge, \vee \rangle$ is a lattice and $\langle A, \cdot, 1 \rangle$ is a monoid such that:

$$a \cdot b \leqslant c \quad \Longleftrightarrow \quad b \leqslant a \backslash c \quad \Longleftrightarrow \quad a \leqslant c/b.$$

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Definition (cf. Galatos et al. 2007)

Let $\mathbf{A}$ be a residuated lattice and $0 \in A$. Then $\langle A, \wedge, \vee, \cdot, 1, 0, \backslash, / \rangle$ is an FL-algebra.

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$$\sim a = a \backslash 0 \qquad \text{and} \qquad -a = 0/a.$$

If $\sim -a = a = - \sim a$ for all $a \in A$, then $\langle A, \wedge, \vee, \cdot, 1, 0, \backslash, / \rangle$ is an InFL-algebra.

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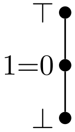
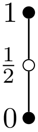
If $\sim -a = a = -\sim a$ for all $a \in A$, then $\langle A, \wedge, \vee, \cdot, 1, 0, \backslash, / \rangle$ is an InFL-algebra. A *distributive* InFL-algebra (DInFL-algebra) has a distributive lattice reduct.

NB: can axiomatize InFL-algebras with $\langle A, \wedge, \vee, \cdot, 1, \sim, - \rangle$.

Examples of DInFL-algebras



$\mathbf{S_3}$



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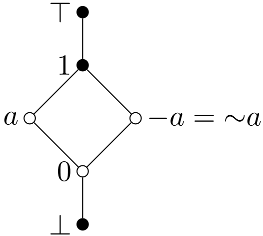
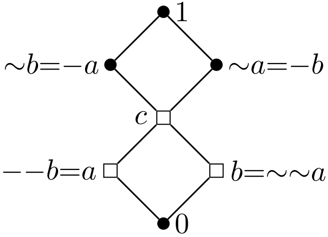
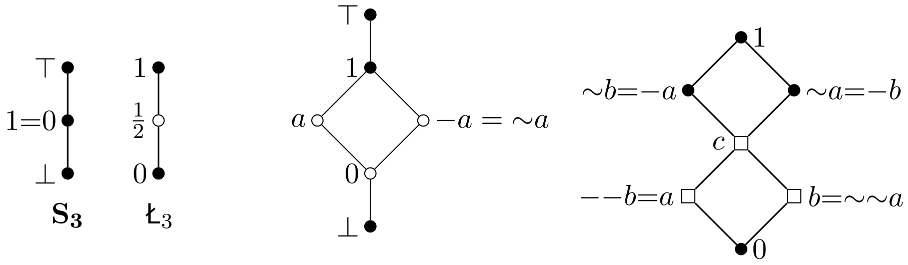




Examples of DInFL-algebras



Example

For $\mathbf{A} = \langle A, \wedge, \vee, ', \bot, \top, \cdot, 1, ^{\sqcup} \rangle$ an RA, let $\sim a = -a = (a')^{\sqcup}$. Then $\langle A, \wedge, \vee, \cdot, 1, \sim, - \rangle$ is a DInFL-algebra.

Quasi relation algebras

Definition (Jipsen & Galatos 2013)

A quasi relation algebra (qRA) is an InFL-algebra with a unary operation $\neg$ satisfying:

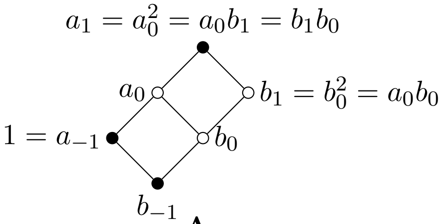
$$\neg(a \vee b) = \neg a \wedge \neg b \qquad \neg\neg a = a$$

$$\neg(a \cdot b) = \sim(-\neg b \cdot -\neg a)$$

A qRA with a distributive lattice reduct will be called a DqRA.

Let $n \in \omega$ with $n \geqslant 3$. If $n = 2k+1$ for $k \geqslant 1$, set

$$A_n = \{a_{-k}, \ldots, a_{-1}, a_0, a_1, \ldots, a_k, b_{-k}, \ldots, b_{-1}, b_0, b_1, \ldots, b_k\}.$$



$\mathbf{A}_3$

Examples of DqRAs

Let $n \in \omega$ with $n \geqslant 3$. If $n = 2k$ for $k \geqslant 2$, set

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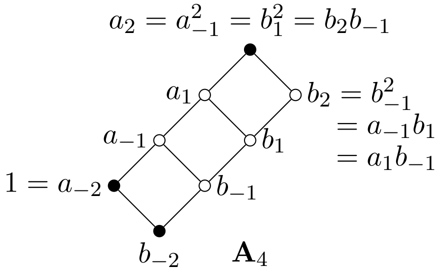

Distributive residuated lattices from pomonoids

Definition

A partially ordered algebra $\mathbf{P} = \langle P, \leqslant, \cdot, 1 \rangle$ is a pomonoid if

- $(P, \leqslant)$ is a poset
- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
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For a pomonoid $\mathbf{P}$ and $U, V \in \mathsf{Up}(P, \leqslant)$ define

$$U \bullet V = {\uparrow}\{\, a \cdot b \mid a \in U, b \in V \,\}.$$

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Theorem (Galatos 2003)

Let $\mathbf{P}$ *be a pomonoid. Then*

$$\mathcal{U}(\mathbf{P}) = \langle \mathsf{Up}(P, \leqslant), \cap, \cup, \bullet, {\uparrow}\{1\}, \backslash, / \rangle$$

is a distributive residuated lattice.

Representing DRLs with binary relations

Consider $\sigma : \mathcal{U}(\mathbf{P}) \to \langle \mathsf{Up}(P^{\partial} \times P), \cap, \cup, ;, \leqslant, \backslash, / \rangle$ defined by

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Let $\langle P, \leqslant, \cdot, 1 \rangle$ be a pomonoid. The map σ preserves meets if and only if whenever $x \cdot u \leqslant y$ & $x \cdot v \leqslant y$, there exists w s.t. $u \leqslant w$, $v \leqslant w$ & $x \cdot w \leqslant y$.

ipo-monoids

Definition (GJKO 2007, GJL 2023)

An *involutive partially ordered monoid* (or *ipo-monoid*) is a structure $\mathbf{P} = \langle P, \leqslant, \cdot, 1, ^{-}, ^{\sim} \rangle$ s.t. $\langle P, \leqslant \rangle$ is a poset, $\langle P, \cdot, 1 \rangle$ is a monoid, and for all $x, y \in P$:

$$x \leqslant y \quad \text{iff} \quad x \cdot y^{\sim} \leqslant 1^{-} \quad \text{iff} \quad y^{-} \cdot x \leqslant 1^{-}.$$

DInFL-algebras from ipo-monoids

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Lemma

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Theorem

Let $\mathbf{P} = \langle P, \leqslant, \cdot, 1, ^{-}, ^{\sim} \rangle$ be an ipo-monoid. Then

$$\mathcal{D}(\mathbf{P}) = \langle \mathsf{Up}(P, \leqslant), \cap, \cup, \bullet, \uparrow 1, -, \sim \rangle$$

is a DInFL-algebra. Moreover, $\mathcal{D}(\mathbf{P})$ is cyclic iff $\mathbf{P}$ is cyclic.

Another nice property of ipo-monoids

Recall $\sigma : \mathcal{U}(\mathbf{P}) \to \langle \mathsf{Up}(P^{\partial} \times P), \cap, \cup, ;, \leqslant, \backslash, / \rangle$ defined by

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Also recall:

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If $\mathbf{P} = \langle P, \leqslant, \cdot, 1, ^-, ^\sim \rangle$ is an ipo-monoid, then it satisfies the condition above.

Proof: $w = x \backslash y$ (since ipo-monoids are residuated).

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Theorem (C., Robinson 2025)

Let $\mathbf{X} = \langle X, \leqslant \rangle$ be a poset and $\alpha : X \to X$ an order automorphism of $\mathbf{X}$. Set $1 = \leqslant$ and for $R \in \mathsf{Up}(X^2, \preccurlyeq)$, define ${\sim}R = R^{c\smile} \mathbin{;} \alpha$ and $-R = \alpha \mathbin{;} R^{c\smile}$. Then

$$\langle \mathsf{Up}(X^2, \preccurlyeq), \cap, \cup, ;, 1, -, \sim \rangle$$

is a DInFL-algebra and it is cyclic iff α is the identity.

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Algebras of the form described above are full DInFL-algebras. Class denoted by FDInFL.

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Definition (C., Robinson 2025)

A DInFL-algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, -, \sim, \rangle$ will be called representable if $\mathbf{A} \in \mathbb{ISP}(\text{FDInFL})$.

DInFL-algebras of binary relations

Theorem

Let $\mathbf{P} = \langle P, \leqslant, \cdot, 1, ^{-}, ^{\sim} \rangle$ *be an ipo-monoid. Then the algebra of binary relations* $\langle \mathsf{Up}\left(P^2, \preccurlyeq\right), \cap, \cup, ;, \leqslant, -, \sim \rangle$ *is a DInFL-algebra.*

Proof.

Define $\alpha : P \to P$ by $\alpha(x) = x^{\sim\sim}$. Then α is an order isomorphism. □

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NB: we can't yet show that $\sigma : \mathsf{Up}(P, \leqslant) \to \mathsf{Up}(P^2, \preccurlyeq)$ is a DInFL-algebra embedding.

Pregroups

Definition (Lambek 1999)

A partially ordered algebra $\langle P, \leqslant, \cdot, 1, ^{\ell}, ^{r} \rangle$ is a pregroup if

- $\langle P, \leqslant, \cdot, 1 \rangle$ is a pomonoid
- $x^{\ell} \cdot x \leqslant 1 \leqslant x \cdot x^{\ell}$ and $x \cdot x^{r} \leqslant 1 \leqslant x^{r} \cdot x$

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Example

Any group $\mathbf{G} = \langle G, \cdot, ^{-1}, e \rangle$ is a pregroup with $\leqslant = \mathrm{id}_G$ and $a^{r} = a^{\ell} = a^{-1}$.

Pregroups

Definition (Lambek 1999)

A partially ordered algebra $\langle P, \leqslant, \cdot, 1, ^\ell, ^r \rangle$ is a pregroup if

- $\langle P, \leqslant, \cdot, 1 \rangle$ is a pomonoid
- $x^\ell \cdot x \leqslant 1 \leqslant x \cdot x^\ell$ and $x \cdot x^r \leqslant 1 \leqslant x^r \cdot x$

Example

Any group $\mathbf{G} = \langle G, \cdot, ^{-1}, e \rangle$ is a pregroup with $\leqslant = \mathrm{id}_G$ and $a^r = a^\ell = a^{-1}$.

Fact: all *finite* pregroups come from finite groups (and hence have $a^r = a^\ell$ and the discrete order).

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Theorem

Let $\mathbf{A}$ be a DInFL-algebra. If there exists a pregroup $\mathbf{P}$ such that $\mathbf{A} \approx \mathcal{D}(\mathbf{P})$, then $\mathbf{A}$ is representable.

Theorem

Let $\mathbf{X} = \langle X, \leqslant \rangle$ be a poset and $\alpha : X \to X$ an order automorphism of $\mathbf{X}$. Set $1 = \leqslant$ and for $R \in \mathsf{Up}(X^2, \preccurlyeq)$, define $\sim R = R^{c\smile} ; \alpha$ and $-R = \alpha ; R^{c\smile}$. Further, if $\beta : X \to X$ is a self-inverse dual order automorphism of $\mathbf{X}$ such that $\beta = \alpha ; \beta ; \alpha$, then defining $\neg R = \alpha ; \beta ; R^c ; \beta$ we get that

$$\langle \mathsf{Up}(X^2, \preccurlyeq), \cap, \cup, ;, 1, -, \sim, \neg \rangle$$

is a distributive quasi relation algebra.

Algebras of the form described above are full DqRAs. Classes denoted by FDqRA.

Definition (C., Robinson 2025)

A DqRA $\mathbf{B} = \langle B, \wedge, \vee, \cdot, 1, -, \sim, \neg \rangle$ is *representable* if $\mathbf{B} \in \mathbb{ISP}\,(\mathsf{FDqRA})$.

Ortho pregroups and DqRAs

Definition (Jipsen & Galatos 2013)

A quasi relation algebra (qRA) is an InFL-algebra with a unary operation $\neg$ satisfying:

$$\neg(a \vee b) = \neg a \wedge \neg b \qquad \neg\neg a = a$$

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Definition

$\mathbf{P} = \langle P, \leqslant, \cdot, 1, {}^{\ell}, {}^{r}, {}^{\neg} \rangle$ is an ortho pregroup if $x^{\neg\neg} = x$ and $xy \leqslant z^{\ell}$ iff $y^{r\neg}x^{r\neg} \leqslant z^{\neg}$.

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Example

If $\mathbf{G} = \langle G, \cdot, ^{-1}, e \rangle$ is a group and $\mathrm{id} : G \to G$ the identity function, then $\langle G, =, \cdot, e, ^{-1}, ^{-1}, \mathrm{id} \rangle$ is an ortho pregroup.

More examples of ortho pregroups

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Let $\mathbf{G} = \langle G, \cdot, ^{-1}, e\rangle$ be an Abelian group. Then $\langle G, =, \cdot, e, ^{-1}, ^{-1}, ^{-1}\rangle$ is an ortho pregroup.

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Recall the Lambek pregroup:

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Consider the set of all unbounded, monotone functions $f : \mathbb{Z} \to \mathbb{Z}$.

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Now let $f^\neg(x) = -f(-x)$. This gives us an ortho pregroup.

Representing DqRAs with binary relations

Theorem

Let $\mathbf{P} = \langle P, \leqslant, \cdot, 1, ^\ell, ^r, ^\neg \rangle$ *be an ortho pregroup. Define* $\neg U = \{ a^\neg \mid a \notin U \}$ *for* $U \in \mathsf{Up}(P, \leqslant)$. *Then*

$$\mathcal{Q}(\mathbf{P}) = \langle \mathsf{Up}(P, \leqslant), \cap, \cup, \bullet, \uparrow\{1\}, \sim, -, \neg \rangle$$

is a DqRA.

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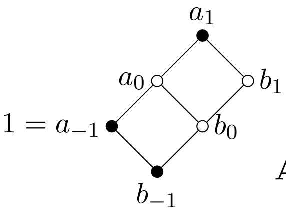
Theorem

Let $\mathbf{A}$ *be a DqRA. If there exists an ortho pregroup* $\mathbf{P}$ *with* $\mathbf{A} \approx \mathcal{Q}(\mathbf{P})$*, then* $\mathbf{A}$ *is representable.*

Proof.

Define $\beta : P \to P$ by $\beta(x) = x^{\neg}$ and use the embedding σ. □

Examples from products of groups



$\mathbf{A}_3$

Theorem

Let $n \geqslant 3$. Then the DqRA $\mathbf{A}_n$ is representable.

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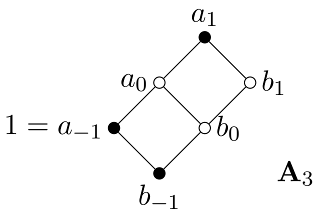

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For $n = 3$, consider the orthopregroup $\mathbf{Z}_7 = \langle \mathbb{Z}_7, =, +, 0, -, -, - \rangle$.
We get $\psi : \mathbf{A}_3 \hookrightarrow \mathcal{Q}(\mathbf{Z}_7)$.

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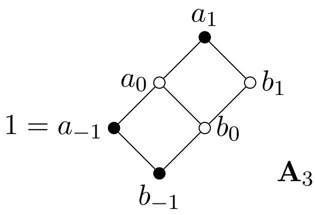

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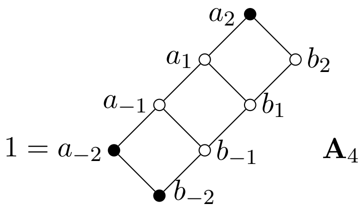
$\psi(b_{-1}) = \emptyset$, $\psi(a_{-1}) = \{0\}$, $\psi(b_0) = \{1, 2, 4\}$, $\psi(b_1) = \mathbb{Z}_7 \backslash \{0\}$ □

Examples from products of groups cont...



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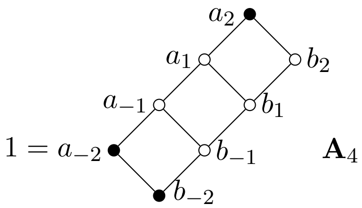
Sketch: For $n \geqslant 4$, consider the ortho pregroup

$$\mathbf{Z}_{n-2} \times \mathbf{Z}_7 = \langle \mathbb{Z}_{n-2} \times \mathbb{Z}_7, =, +, (0,0), -, -, - \rangle.$$

$$T = \{(0,1), (0,2), (0,4)\} \cup \{(m,\ell) \mid 1 \leqslant m \leqslant n-3, \ell \in \{3,5,6\}\}.$$

Use T to define the images of the b_i under an embedding $\psi : A_n \to \mathsf{Up}(\mathbb{Z}_{n-2} \times \mathbb{Z}_7, =)$.

Examples from products of groups cont...



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Use T to define the images of the b_i under an embedding $\psi : A_n \to \mathsf{Up}(\mathbb{Z}_{n-2} \times \mathbb{Z}_7, =)$. Then, let $\psi(a_i) = \psi(b_i) \cup \{(0,0)\}$. □

Ongoing work

- Describe possible $^\neg$ operations on those pregroups which come from groups. This would lead to a description of all possible finite ortho pregroups.
- The condition required for σ to preserve meets is also satisfied by lattice-ordered monoids. What do we know about the residuated lattice $\langle \mathsf{Up}(P, \leqslant), \cap, \cup, \bullet, \uparrow\{1\}, \backslash, / \rangle$ when $\mathbf{P}$ is an ℓ-monoid?
- Extend to (first define!) pregroupoids to emulate Brandt groupoids of Jónsson and Tarski. See also Jipsen (2017).
- Adapt the methods of Andréka, Nemeti & Givant (2020) to find *non-representable* DInFL-algebras & DqRAs.

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