Simulation Quotients Simulating Bisimulation Quotients

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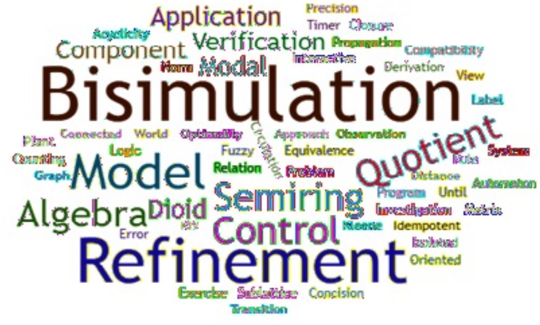
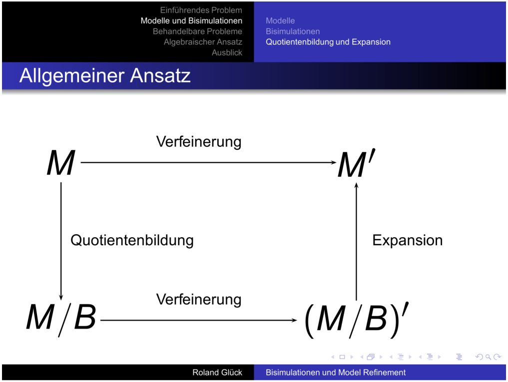
Recent Work

- 2008: Circulations, Fuzzy Relations and Semirings
- 2009: A Semiring Approach to Equivalences, Bisimulations and Control
- 2010: Model Refinement Using Bisimulation Quotients
- 2011: Using Bisimulations for Optimality Problems in Model Refinement
- 2012: Two Observations in Dioid Based Model Refinement
- 2014: Bisimulations and Model Refinement
- 2014: Exploring Modal Worlds
- 2015: Towards Interactive Verification of Programmable Logic Controllers Using Modal Kleene Algebra and KIV
- 2017: Algebraic Investigation of Connected Components
- 2018: Distances, Norms and Error Propagation in Idempotent Semirings
- 2018: Algebraic Derivation of Until Rules and Application to Timer Verification
- 2021: Isolated Sublattices and their Application to Counting Closure Operators
- 2022: Components and acyclicity of graphs. An exercise in combining precision with concision
- 2023: Compatibility of Refining and Controlling Plant Automata with Bisimulation Quotients
- 2024: A Matrix-Oriented View of Bisimulation Quotients over Dioid-Labeled Transition Systems

Main Topics


Application
Precision
Timer
Component
Verification
Derivation
View
Modal
Bisimulation
Label
Connected
World
Observation
Quotient
Logic
Fuzzy
Equivalence
Model
Relation
Distance
Automaton
Semiring
Program
Until
Algebra
Dioid
Error
Control
Idempotent
Refinement
Oriented
Concision
Transition

Idea


Einführendes Problem
Modelle und Bisimulationen
Behandelbare Probleme
Algebraischer Ansatz
Ausblick
Modelle
Bisimulationen
Quotientenbildung und Expansion
Allgemeiner Ansatz
M
Verfeinerung
M′
Quotientenbildung
Expansion
M/B
Verfeinerung
(M/B)′
Roland Glück
Bisimulationen und Model Refinement

Generalization

- model: already generic topic (comprising automata, transition systems, labeled graphs, ...)
- control/refinement: also generic
- bisimulation: room for generalization
- what about simulations?

Models

Definition

A *model* is a structure $M = (V, E, a)$ where (V, E) is a directed graph, $a : V \to 2^{\Pi}$ is the *node labeling* function, and $s \in V$ is the unique *initial* or *start node* with $\mathrm{I} \in a(s)$.

- *live model*: every walk starting at s can be prolonged (no dead-ends)
- *reduced model*: except possibly s, no node without incoming edges has an outgoing edge (no ghost streets)
- *normalized model*: both live and reduced model

Satisfaction and Refinement

- *M satisfies* Φ, denoted by $M \models_r \Phi$, if M is live and $s \models \Phi$.
- $M' = (V', E', a')$ is called a *refinement* of $M = (V, E, a)$ if $V' = V$, $E' \subseteq E$, and $a' = a$.
- notation $M' \leq M$
- goal: given M and Φ, find $M' \leq M$ with $M' \models_r \Phi$

Theorem

$M \models_r \Phi \Leftrightarrow$ *there is a normalized refinement* $M' \leq M$ *with* $M' \models_r \Phi$.

Simulations

Definition

Let $M_1 = (V_1, E_1, a_1)$ and $M_2 = (V_2, E_2, a_2)$ be two models. A left-total relation $\mathcal{S} \in V_1 \times V_2$ is called an (M_1, M_2)-*simulation* if it fulfills the following two properties for all u_1, u_2, v_1, and v_2:

- $(v_1, v_2) \in \mathcal{S} \Rightarrow a_1(v_1) = a_2(v_2)$
- $(u_1, v_1) \in E_1 \wedge (u_1, u_2) \in \mathcal{S} \Rightarrow \exists v_2 : (v_1, v_2) \in \mathcal{S} \wedge (u_2, v_2) \in E_2$

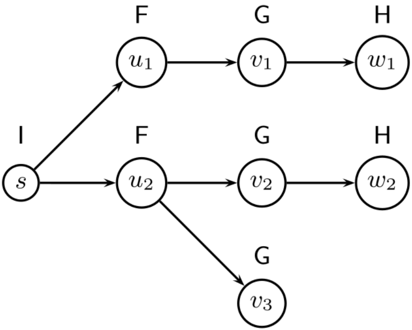
Notation:

- $M_1 \preceq_{\mathcal{S}} M_2$: $\mathcal{S}$ is an (M_1, M_2)-simulation
- $M_1 \preceq M_2$: $\exists \mathcal{S} : M_1 \preceq_{\mathcal{S}} M_2$
- $M_1 \simeq M_2$: $M_1 \preceq M_2 \wedge M_2 \preceq M_1$ (simulation equivalence, !!!ATTENTION!!!)

Autosimulations and Simulation Order

- *autosimulation* for M: (M, M)-simulation
- $\mathcal{I}$ is autosimulation
- closed under composition
- $v \preceq w \Leftrightarrow_{df} \exists$ autosimulation $\mathcal{S}$ with $(v, w) \in \mathcal{S}$
- $\preceq$ is greatest autosimulation
- $\preceq$ is preorder
- $v \simeq w \Leftrightarrow_{df} v \preceq w \wedge w \preceq v$
- $\simeq$ is an equivalence

Example



Quotients

Definition

Let $M = (V, E, a)$ be a model. Then we define the *simulation quotient* $M/\simeq =_{def} (V/\simeq, E/\simeq, a/\simeq)$ by

- $V/\simeq =_{def} [V]_\simeq$
- $([u]_\simeq, [v]_\simeq) \in E/\simeq \Leftrightarrow_{df} \exists u' \in [u]_\simeq, v' \in [v]_\simeq : (u', v') \in E$
- $(a/\simeq)([v]_\simeq) =_{def} a(v)$

- $M/\simeq$ and M are simulation equivalent
- for submodels:
  $([u]_\simeq, [v]_\simeq) \in M'/\simeq \Leftrightarrow_{df} \exists u' \in [u]_\simeq, v' \in [v]_\simeq : (u', v') \in M'$
- idea here: instead of constructing $M' \leq M$ with $M' \models_r \Phi$, construct first $(M/\simeq)' \leq M$ with $M/\simeq \models_r \Phi$ and derive susequently M' from $(M/\simeq)'$

Expansion

Definition

Let M be a model, and consider a refinement $(M/\simeq)'$ of its simulation quotient $M/\simeq$. Then we define the *expansion* $(M/\simeq)'\backslash\simeq$ as the refinement of M with the edge set given by

$$(\hat{u},\hat{v}) \in (M/\simeq)'\backslash\simeq \Leftrightarrow_{df}$$
$$(\hat{u},\hat{v}) \in M \wedge \exists([u']_{\simeq},[v']_{\simeq}) \in (M/\simeq)' : \hat{u} \in [u']_{\simeq} \wedge \hat{v} \in [v']_{\simeq}$$

- edges in $(M/\simeq)'$ serve as indicators which edges from M to keep in $(M/\simeq)'\backslash\simeq$

Simulation Quotients vs. Bisimulation Quotients

- bisimulations: $([u]_\sim, [v]_\sim) \in M/\sim \; \Rightarrow \exists (u, v') \in M$ with $v' \in [v]_\sim$
- simulations: $([u]_\simeq, [v]_\simeq) \in M/\simeq \; \Rightarrow \exists (u, v') \in M$ with $v \preceq v'$
- walks in $(M/\simeq)'$ are pushed upwards (wrt. $\preceq$) in the expansion

Simulation Up-Closure

Definition

For a refinement $M' \leq M$, the *simulation up-closure* $sucl(M')$ is given by

$$(u,v) \in sucl(M') \Leftrightarrow_{df} (u,v) \in M \wedge \exists u', v' : u' \preceq u \wedge v' \preceq v \wedge (u', v') \in M'$$

Φ is *simulation up-closed* if for every normalized refinement $M' \leq M$ with $M' \models_r \Phi$ also $sucl(M') \models_r \Phi$ holds.

- $sucl(M')$ can be seen as the most general refinement of M with the same behavior
- F, $\bigcirc$F, and $\Box$F are simulation up-closed
- $\bigcirc\bigcirc\bigcirc$F is not simulation up-closed

Refinement via Quotients

Definition

Φ is *compatible with simulation equivalence* if for all $M_1 \simeq M_2$ the equivalence $M_1 \models_r \Phi \Leftrightarrow M_2 \models_r \Phi$ holds.

- examples: $\forall$CTL, $\forall$CTL* (containing LTL)

Definition

M is called *simulation dominated* if for every $(u, v) \in M$ there is a $\preceq$-maximal $v' \in succ(u)$ with $(u, v') \in M$.

- holds in particular for finite models

Theorem

Let Φ be a simulation up-closed temporal formula compatible with simulation equivalence, and let M be a simulation dominated model. Then M is refineable w.r.t. Φ iff $M/\simeq$ is refineable w.r.t. Φ.

Dioids

Definition

A *dioid* is a structure $(\Delta, \oplus, 0, \otimes, 1)$ such that

- $(\Delta, \oplus, 0)$ is a commutative and idempotent monoid,
- $(\Delta, \otimes, 1)$ is a monoid,
- 0 is both a left and right annihilator of $\otimes$, and
- $\otimes$ distributes both from left and right over $\oplus$.

- order defined by $x \sqsubseteq y =_{def} x \oplus y = y$
- used for shortest/maximum capacity paths
- automata theory
- and other topics

Matrices over Dioids

- matrices over dioids correspond to (complete) labeled graphs
- 0-1-matrices are isomorphic to relations
- equivalence decomposition: E is equivalence $\Rightarrow \exists$ right-total function D with $DD^t = E$
- reflexivity, transitivity etc. definable by algebraic means
- relational simulation: $G_1 \preceq_S G_2 \Leftrightarrow S^\circ; G_1 \subseteq G_2; S^\circ$ and S is left-total

Definition

Let $A \in \Delta^{m\times m}$ and $B \in \Delta^{n\times n}$ be two quadratic Δ-matrices, and let $S \in \{0,1\}^{n\times m}$ be a right-total 0-1-matrix. S is called an (A, B)-*simulation* if $SA \sqsubseteq BS$ in which case we write $A \preceq_S B$.

Linear Fixpoints

- same constructions as in traditional setting (simulation equivalence etc.)
- $A/D =_{def} D^t AD$, $b/D =_{def} D^t b$, $\hat{b}\backslash D =_{def} D\hat{b}$
- interest in fixpoints of $f(x) = Ax + b$
- b corresponds to (initial) node labeling
- b compatible with $A \Leftrightarrow Ab = b$

Theorem

Let S be a simulation preorder of $A \in \Delta^{n\times n}$, let $D \in \Delta^{n\times m}$ be an equivalence decomposition of E_S, and let b be a column vector compatible with S. Assume that x_μ is the least solution of the equation $Ax \oplus b = x$, and that $\hat{x}_\mu$ is the least solution of $(A/D)\hat{x} \oplus (b/D) = \hat{x}$. Then we have $x_\mu = \hat{x}_\mu \backslash D$.

Open Questions

- syntactic/logical characterization of simulation up-closeness of Φ
- computation of the greatest dioid-based matrix simulation