On the Reachability Problem on Monoid-Labelled Undirected Graphs

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Joint work with Harshil Mittal and Jayalal Sarma

Indian Institute of Technology, Madras

22nd International Conference on
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9th April 2026

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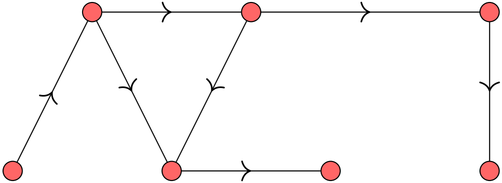

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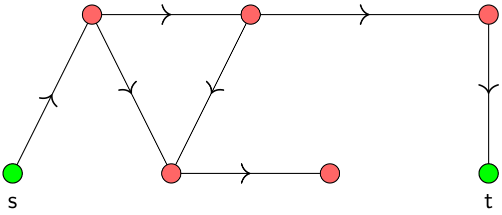

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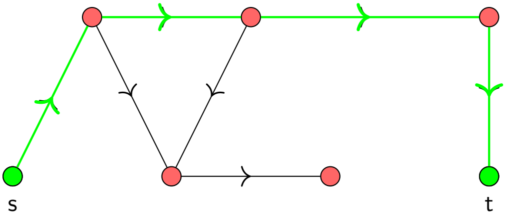

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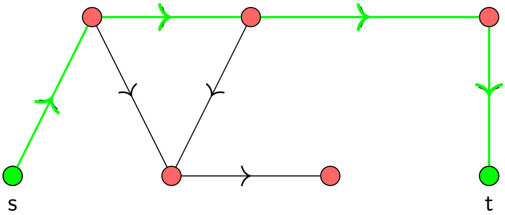

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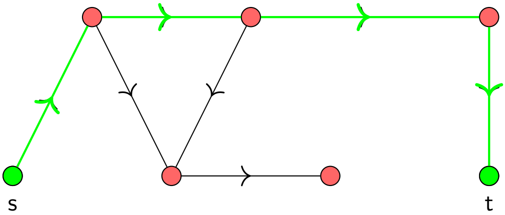

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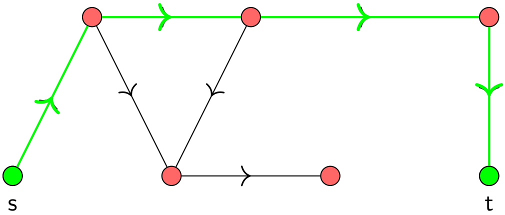

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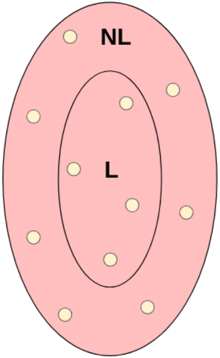

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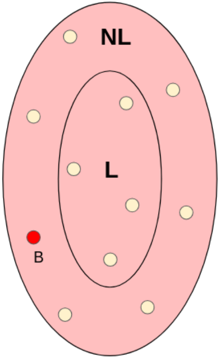
- A problem B is NL-complete if it is in NL and every problem in NL can be reduced to B in logspace.



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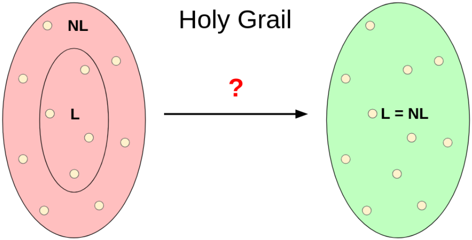

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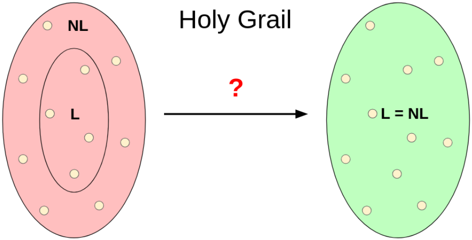

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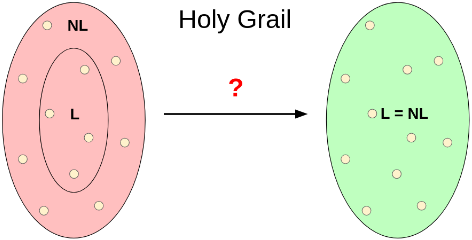

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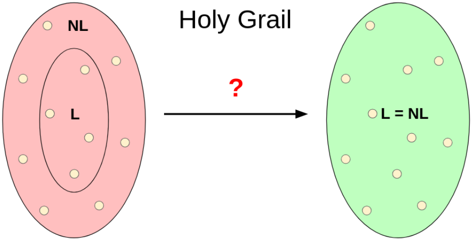

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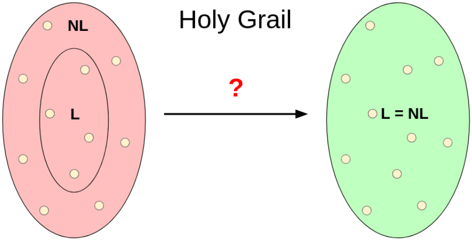

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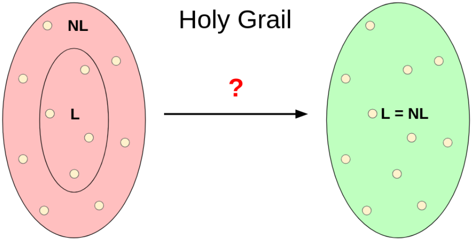

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- Directed Reachability ($\textsc{Reach}$) is NL-complete.

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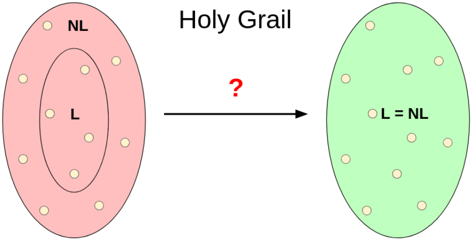

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- $\overline{\mathrm{REACH}}$ is NL-complete [Imm88, Sze88].

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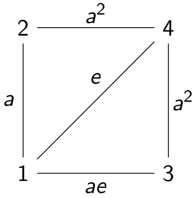

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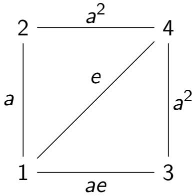

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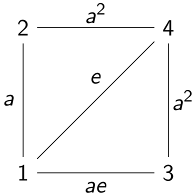

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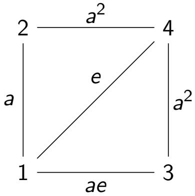

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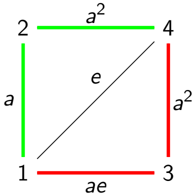

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α	0	α	0	$\alpha\beta$	0	α
β	0	β	$\beta\alpha$	0	β	0
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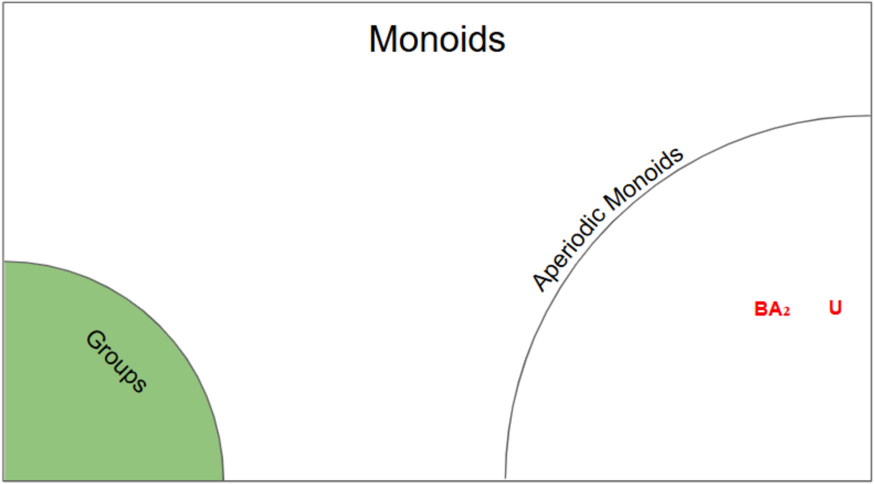
Into the World of Monoids



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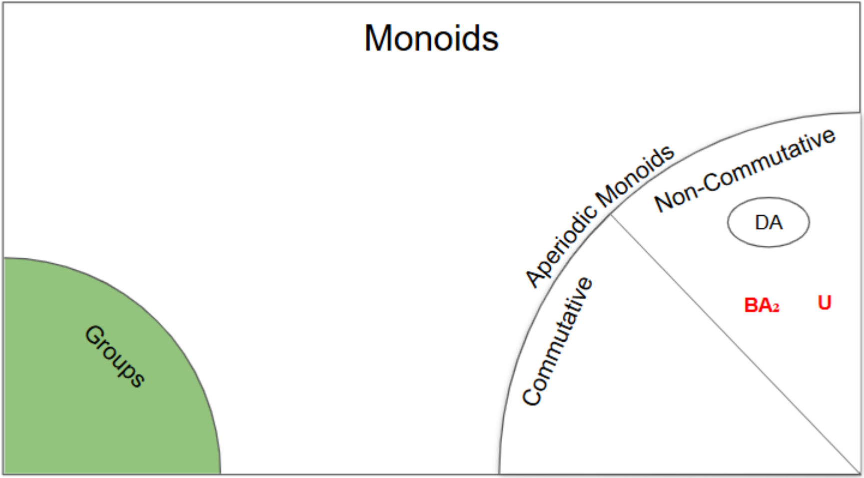

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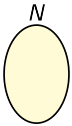

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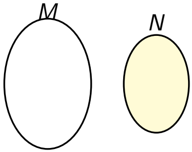

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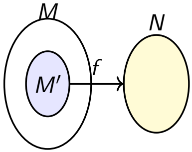

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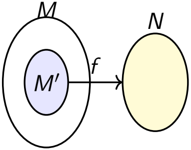
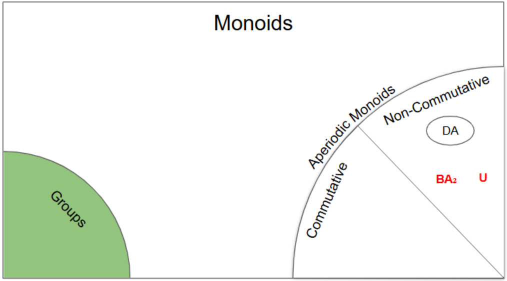


Proposition [Pin86, Tes98]

Any non-commutative aperiodic monoid, that is not in the class DA is divided by either BA_2 or U.



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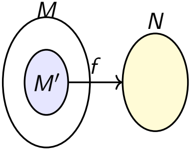

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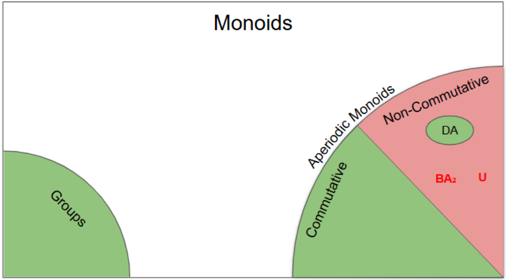
Dichotomy Theorem [RSS19]

Let M be an aperiodic monoid. Then, $\textsc{Reach}_M$ is either in logspace or is NL-hard.



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Refined study on reachability over various monoid classes with different accepting sets.

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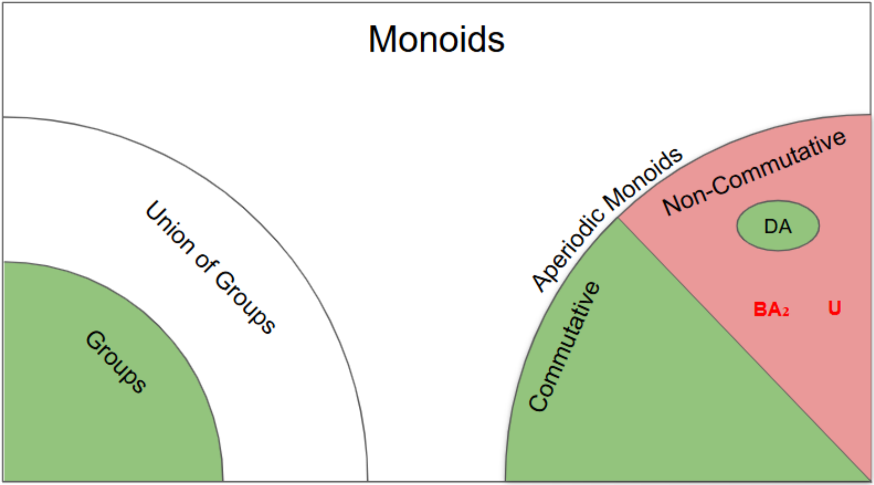
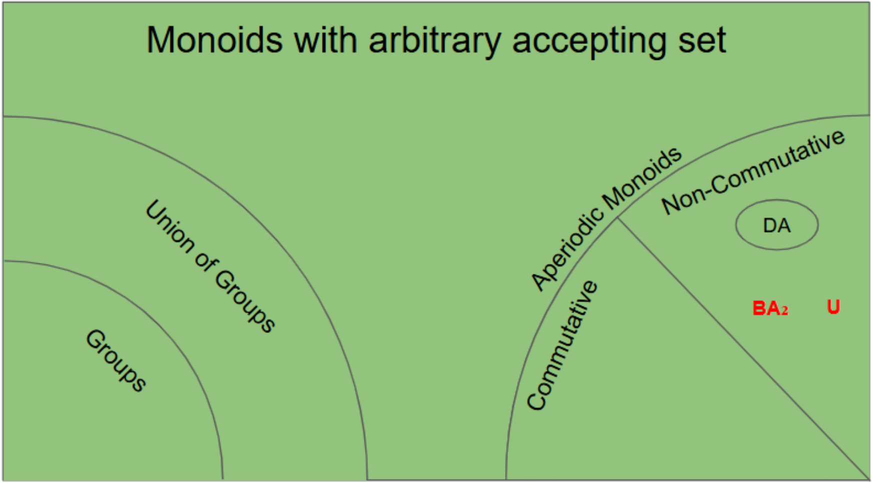


Ideal Classification



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Theorem

Let M be a commutative monoid, with $F \subseteq M$. $\textsc{Reach}_{M,F}$ is in logspace.

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Dichotomy for BA_2, U with Arbitrary Accepting Set

Recall: Let M be any monoid, with $F = \{id\}$ or $F = M \setminus \{id\}$. Then, $\textsc{Reach}_{M,F}$ is in logspace.

Dichotomy for $\mathrm{BA}_2, \mathrm{U}$ with Arbitrary Accepting Set

Recall: Let M be any monoid, with $F = \{id\}$ or $F = M \setminus \{id\}$. Then, $\textsc{Reach}_{M,F}$ is in logspace.

Theorem

Let M be BA_2 (or U respectively), with $F \subset \mathrm{BA}_2 \setminus \{\mathrm{id}\}$ (or $F \subset \mathrm{U} \setminus \{\mathrm{id}\}$). Then, $\textsc{Reach}_{M,F}$ is NL-hard.

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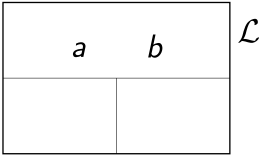

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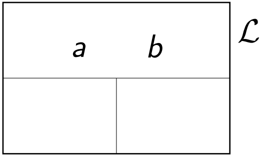

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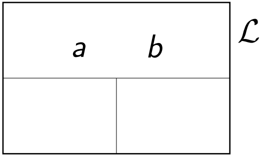

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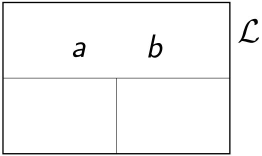

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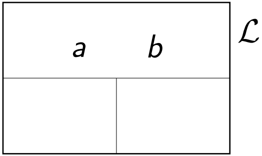

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Proposition

In a monoid M, a $\mathcal{H}$-class H contains an idempotent element if and only if H forms a maximal subgroup of M.

Union of Groups (UoG) Monoid

Union of Groups Monoid [Cli41]

M is a union of groups (completely regular) monoids if each $\mathcal{H}$-class contains an idempotent element and thus forms a maximal subgroup of M.

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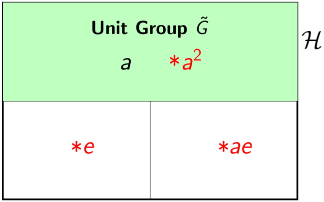
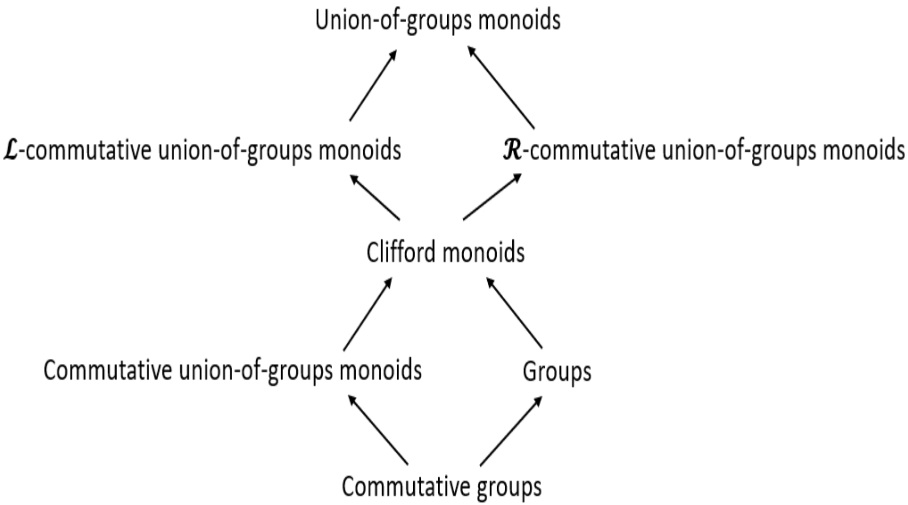


Union of Groups Monoids Classification



Source: this work

Product Graph [RSS19]

Consider a graph $G = (V, E)$, with $\phi : E \to M$. An unlabelled digraph $G' = (V', E')$ is the *product graph* of G, such that $V' = V \times M$ and,

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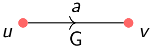

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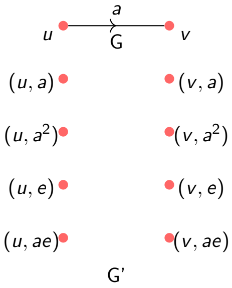

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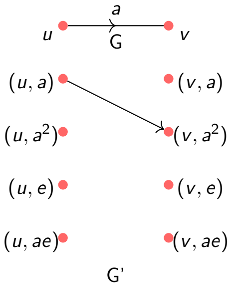

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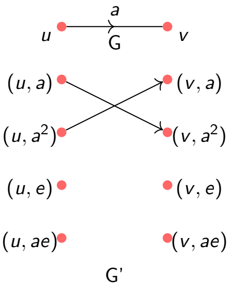

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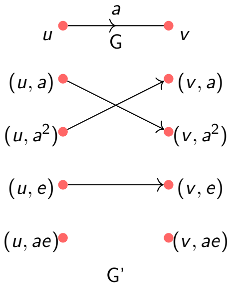

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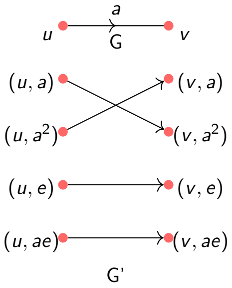

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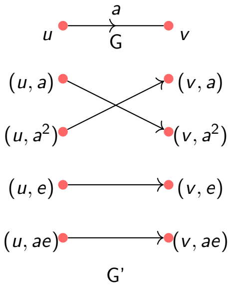

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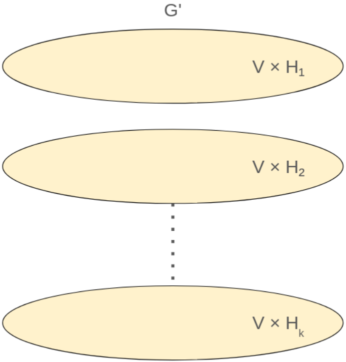

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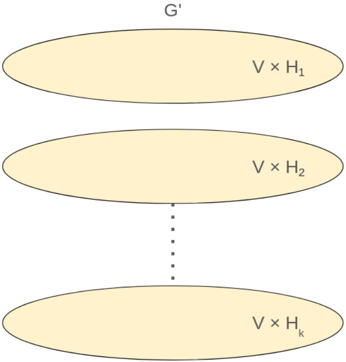

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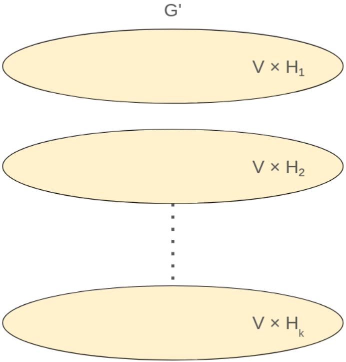

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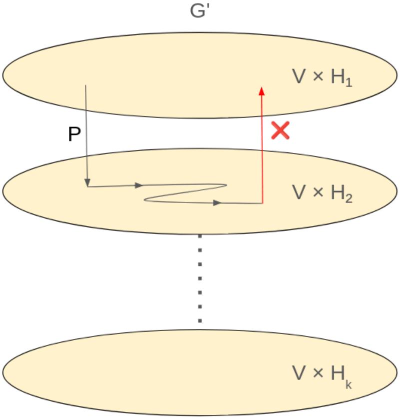

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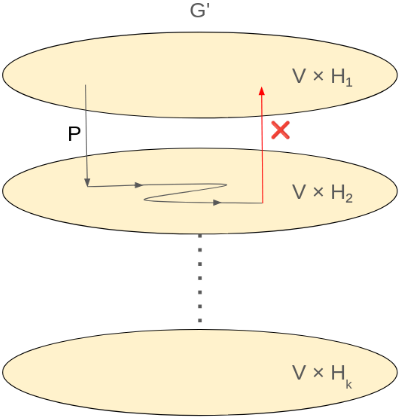
- From above lemma, path P must be such that for every $\mathcal{H}$-class H, it enters/leaves the part $V \times H$ at most once.



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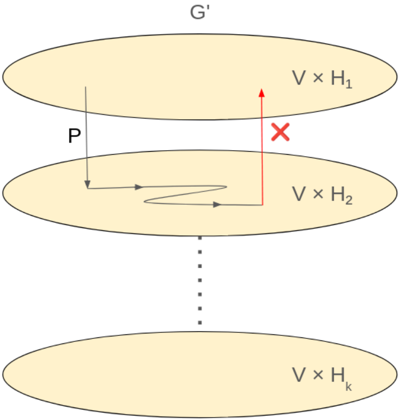

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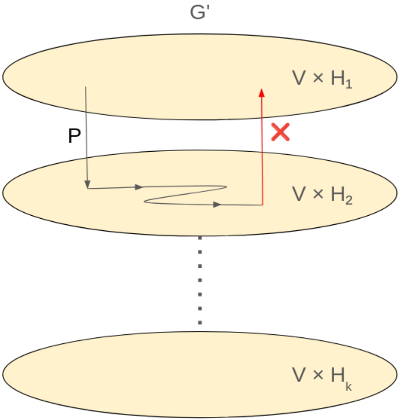
- Guess the $\mathcal{H}$-classes H whose corresponding $V \times H$ parts are visited and guess the edges involved in the jumps across these parts.
- Number of $\mathcal{H}$-classes is $\mathcal{O}(1)$, there are only $n^{\mathcal{O}(1)}$ guesses.



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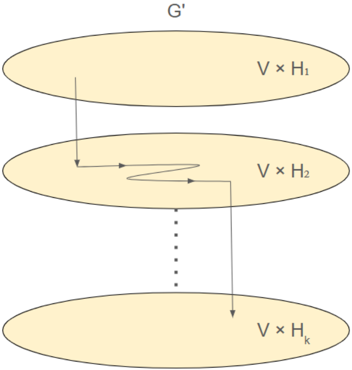
- Task: Check for each visited part $V \times H$, whether there is a path in $G'[V \times H]$ from the vertex at which the guessed edge entering into $V \times H$ lands, to the vertex at which the guessed edge exiting $V \times H$ departs from.



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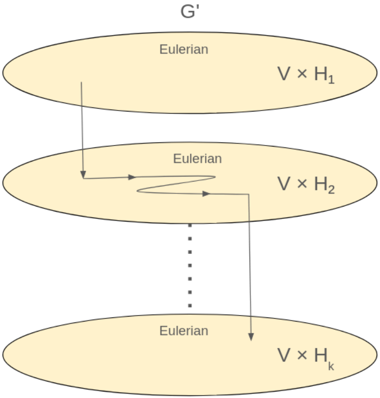

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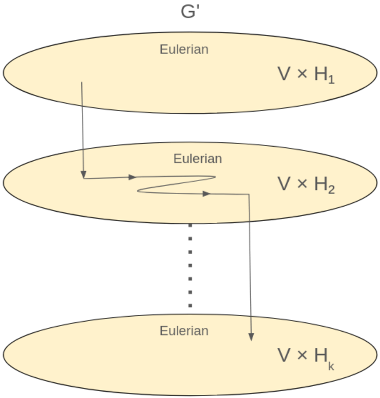

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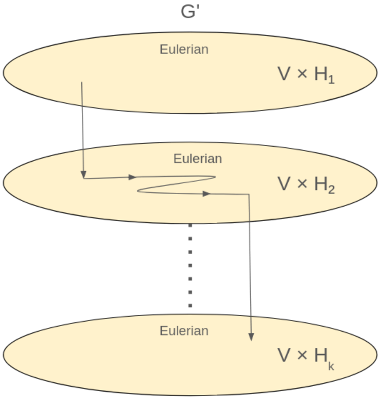
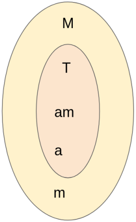
- Eulerian graph is a directed graph with the indegree of each vertex in the graph is equal to its outdegree.
- Use the logspace algorithm for Eulerian directed reachability [RTV06] for each $G'[V \times H]$.



Restricted UoG Monoid with Restricted Accepting Set

$T \subseteq M$ is said to be *right ideal* of M if for all $a \in T$ and $m \in M$, $am \in T$.



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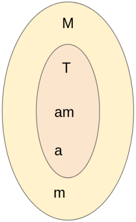
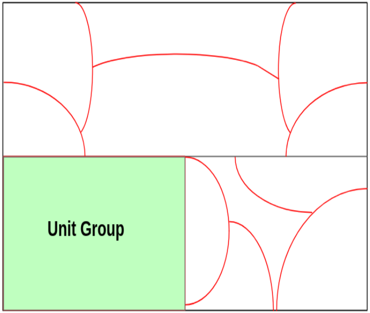


Theorem

Let M be a Union of Groups monoid such that all $\mathcal{R}$-classes other than the unit group (say $\tilde{G}$) are right ideals.



$\mathcal{R}$

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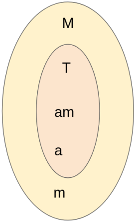

Theorem

Let M be a Union of Groups monoid such that all $\mathcal{R}$-classes other than the unit group (say $\tilde{G}$) are right ideals. Then, $\text{REACH}_{M,F}$ is in logspace when the accepting set F is any of the $\mathcal{R}$-classes of M other than the unit group $\tilde{G}$.

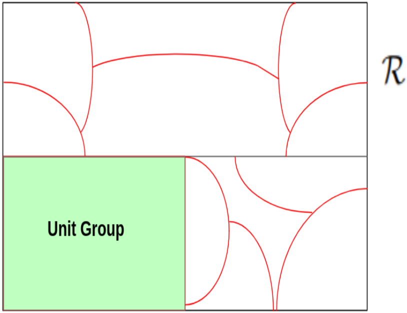

Table of Contents

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- Study revealed finer complexity upper and lower bounds, and dichotomies using the monoid structure and accepting sets.

Summary of our Results

Monoid M	**Accepting set** F	**Complexity of** $\textsc{Reach}_{M,F}$
Any Monoid	$F \subseteq$ unit group $\tilde{G}$ or $F = M \setminus \tilde{G}$	L
Commutative	$F \subseteq M$	L
$\mathcal{L}$ (or $\mathcal{R}$)-commutative union of groups	$F \subseteq M$	L
Union of groups where every $\mathcal{R}$-class other than unit group is right ideal	$F \subseteq$ any $\mathcal{R}$-class of M other than unit group	L
BA_2 or U	$F \subset M \setminus \{\text{identity}\}$	NL-hard
Monoid with absorber 0 and an idempotent $\mu = x \cdot y \neq 0$ such that $x^2 = 0$ or $y^2 = 0$	$F = \{\mu\}$	NL-hard

THANK YOU

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