On the Dual Composition of Relations

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Motivation I

My personal motivation to investigate situations in which the collection of complemented relations is closed under composition. This question arose from a situation encountered in the relation algebraic definition of the real numbers. In this context, the least upper bound property of the reals becomes:

Theorem (Least-Upper-Bound Property)

For every relation $X : A \to \mathbb{R}$ *we have* $\mathrm{dom}(X) \sqcap \mathrm{dom}(\mathrm{ubd}_E(X)) \sqsubseteq \mathrm{dom}(\mathrm{lub}_E(X))$

Motivation II

Lemma

If $X;C^{\smile}$ *is regular, then we have* $\mathrm{dom}(X) \sqcap \mathrm{dom}(\mathrm{ubd}_E(X)) = \mathrm{dom}(\mathrm{lub}_E(X))$.

where C is the linear strict-order on the reals and a relation R is regular if $R = \neg\neg R$.

Motivation II

Lemma

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where C is the linear strict-order on the reals and a relation R is regular if $R = \neg\neg R$. We actually have

1. If R is complemented, then R is regular.
2. C is complemented.
3. If X is complemented and complemented relations are closed under composition, then $X ; C^{\smile}$ is regular.

Motivation III

Examples:

1. Boolean valued relations, i.e., relations of the form $A \times B \to \mathcal{B}$ with a Boolean algebra $\mathcal{B}$ are closed under composition.
2. Fuzzy relations (over the unit interval) that are complemented are closed under composition.

These examples have in common that the reversed order of the lattice of truth values is the same lattice again. Therefore, any operation on relations has also its dual version. In the case of relation algebras, the dual version of composition is known as the relative sum. For concrete relations it is given by:

$$Q \mathbin{\dagger} R = \{(a,c) \mid \forall b : (a,b) \in Q \vee (b,c) \in R\}.$$

The fact that ; and $\dagger$ are dual to each other is expressed by the equations

$$Q \mathbin{\dagger} R = \overline{\overline{Q} ; \overline{R}}, \qquad Q ; R = \overline{\overline{Q} \mathbin{\dagger} \overline{R}}.$$

Distributive allegories

Definition

A distributive allegory $\mathcal{R} = \langle \mathrm{Obj}_{\mathcal{R}}, \mathrm{Mor}_{\mathcal{R}}, ;, \mathbb{I}, \sqcap, \sqcup, \perp\!\!\!\perp, \breve{} \rangle$ is a category satisfying

1. for all objects A and B the collection $\mathcal{R}[A,B]$ is a distributive lattice with binary meet $\sqcap$, binary join $\sqcup$, induced order $\sqsubseteq$, and least element $\perp\!\!\!\perp_{AB}$,
2. $Q;\perp\!\!\!\perp_{BC} = \perp\!\!\!\perp_{AC}$ for all relations $Q : A \to B$,
3. $Q\breve{}\breve{} = Q$ for all $Q : A \to B$,
4. $(Q \sqcap R)\breve{} = Q\breve{} \sqcap R\breve{}$ for all $Q, R : A \to B$,
5. $(Q;R)\breve{} = R\breve{};Q\breve{}$ for all $Q : A \to B$ and $R : B \to C$,
6. $Q;(R \sqcap S) \sqsubseteq Q;R \sqcap Q;S$ for all $Q : A \to B$ and $R, S : B \to C$,
7. $Q;(R \sqcup S) = Q;R \sqcup Q;S$ for all $Q : A \to B$ and $R, S : B \to C$,
8. for all relations $Q : A \to B$, $R : B \to C$ and $S : A \to C$, the modular inclusion $Q;R \sqcap S \sqsubseteq Q;(R \sqcap Q\breve{};S)$ holds.

p-allegories

Definition

A pseudo-complemented distributive allegory (or a p-allegory for short) is a structure $\mathcal{R} = \langle \mathrm{Obj}_\mathcal{R}, \mathrm{Mor}_\mathcal{R}, ;, \mathbb{I}, \sqcap, \sqcup, \neg, \perp\!\!\!\perp, \breve{} \rangle$ so that $\langle \mathrm{Obj}_\mathcal{R}, \mathrm{Mor}_\mathcal{R}, ;, \mathbb{I}, \sqcap, \sqcup, \perp\!\!\!\perp, \breve{} \rangle$ is a distributive allegory and for all $Q, R : A \to B$ we have

$$Q \sqcap R \sqsubseteq \perp\!\!\!\perp_{AB} \text{ iff } Q \sqsubseteq \neg R.$$

Following usual conventions we call a relation $R : A \to B$ a complement of $Q : A \to B$ iff $Q \sqcap R = \perp\!\!\!\perp_{AB}$ and $Q \sqcup R = \top\!\!\!\top_{AB}$. Q is called complemented iff it has a complement.

Definition

A Heyting allegory is a distributive allegory satisfying

1. $Q \sqcap R \sqsubseteq S$ iff $Q \sqsubseteq R \to S$ for all $Q, R, S : A \to B$,
2. $Q ; R \sqsubseteq S$ iff $Q \sqsubseteq S/R$ for all $Q : A \to B$, $R : B \to C$ and $S : A \to C$.

Example

Consider the set $\mathrm{At} = \{1', a, b, c\}$ and its power set $\mathcal{P}(\mathrm{At})$, i.e., the set of all subsets of At. Using Maddux's notation the relation algebra 37_{65} (distributive allegory with exactly one object) is defined by the following operations on atoms:

;	$1'$	a	b	c
$1'$	$\{1'\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{1', a, b\}$	$\{a, b, c\}$	$\{b, c\}$
b	$\{b\}$	$\{a, b, c\}$	$\{1', a, b\}$	$\{a, c\}$
c	$\{c\}$	$\{b, c\}$	$\{a, c\}$	$\{1', a, b\}$

x	$x^{\smile}$
$1'$	$1'$
a	a
b	b
c	c

We consider the subset $A = \mathcal{P}(\mathrm{At}) \setminus \{\{c\}, \{a, c\}, \{1', c\}, \{1', a, c\}\}$. It is easy to verify that this subset is closed under all operations of the relation algebra except complementation, and is, therefore, also a distributive allegory. Since it is finite, it is also a p-allegory.

$\{a\}$ is complemented (in A) because $\{1', b, c\} \in A$. Furthermore, we obtain

$$\{a\};\{a\} = a;a = \{1', a, b\}.$$

Since $\{c\} \notin A$, the relation $\{a\};\{a\}$ is not complemented.

Duplex distributive allegories

Definition

A duplex distributive allegory $\mathcal{R}$ is a distributive allegory and $\mathcal{R}$ with the operations $\dagger$, $\mathbb{D}$ and the reversed order structure is distributive allegory so that

1. For all relations $Q : A \to B$, $R : B \to C$ and $S : C \to D$, the mixed lax associativity law $Q;(R \dagger S) \sqsubseteq (Q;R) \dagger S$ holds.
2. $\mathbb{I}_A$ and $\mathbb{D}_A$ are complements, i.e., we have $\mathbb{I}_A \sqcap \mathbb{D}_A = \perp\!\!\!\perp_{AA}$ and $\mathbb{I}_A \sqcup \mathbb{D}_A = \top\!\!\!\top_{AA}$.

The duplex allegory Rel(L)

Given a complete Heyting algebra L, an L-fuzzy relation Q from a set A to a set B is a function $Q : A \times B \to L$, i.e., Q is a characteristic function of an L-fuzzy subset of $A \times B$. It is well-known that Rel(L) with sets as objects and L-fuzzy relations as morphisms forms a Heyting allegory.
A complete double Heyting algebra is a complete Heyting algebra so that the structure by reversing its order is also a complete Heyting algebra.

Theorem

Let L be a complete double Heyting algebra, then Rel(L) *together with the dual composition*

$$(Q \,\dot{;}\, R)(a,c) = \bigsqcap_{b \in B} Q(a,b) \sqcup R(b,c) \text{ and } \mathbb{D}(a,b) = \begin{cases} 0 & \text{if } a = b, \\ 1 & \text{otherwise.} \end{cases}$$

is a duplex allegory.

Ideals and dual ideals I

Definition

Let $\mathcal{R}$ be a p-allegory and $Q : A \to B$. Then $Q : A \to B$ is called a right ideal iff $Q ; \Pi_{BB} = Q$, a left ideal iff $\Pi_{AA} ; Q = Q$, and an ideal iff $\Pi_{AA} ; Q ; \Pi_{BB} = Q$.

The collection of all ideal relations between given sets is in a one-to-one correspondence to the elements of L, justifying that ideals can serve as an abstract version of the truth values used by the relations in $\mathrm{Rel}(L)$.

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Definition

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The collection of all ideal relations between given sets is in a one-to-one correspondence to the elements of L, justifying that ideals can serve as an abstract version of the truth values used by the relations in $\mathrm{Rel}(L)$.

Definition

Let $\mathcal{R}$ be a duplex allegory. A relation $Q : A \to B$ is called a right dual ideal iff $Q \,\dot{;}\, \perp\!\!\perp_{BB} = Q$, a left dual ideal iff $\perp\!\!\perp_{AA} \,\dot{;}\, Q = Q$, and a dual ideal iff $\perp\!\!\perp_{AA} \,\dot{;}\, Q \,\dot{;}\, \perp\!\!\perp_{BB} = Q$.

Ideals and dual ideals II

Theorem

Let $\mathcal{R}$ be a duplex allegory. Then we have

1. *Q is a right ideal iff Q is a right dual ideal,*
2. *Q is a left ideal iff Q is a left dual ideal,*
3. *Q is an ideal iff Q is a dual ideal.*

Duplex p-allegories I

Definition

A duplex pseudo-complemented distributive allegory (duplex p-allegory) is a structure consisting of two p-allegories, i.e., $\langle \mathrm{Obj}_{\mathcal{R}}, \mathrm{Mor}_{\mathcal{R}}, ;, \mathbb{I}, \sqcap, \sqcup, \neg, \perp\!\!\!\perp, \breve{} \rangle$ and $\langle \mathrm{Obj}_{\mathcal{R}}, \mathrm{Mor}_{\mathcal{R}}, \dagger, \mathbb{D}, \sqcup, \sqcap, \sim, \top\!\!\!\top, \breve{} \rangle$ are p-allegories.

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Theorem

Let $\mathcal{R}$ be a duplex p-allegory and $Q : A \to B$ and $R : B \to C$. Then we have

1. *if Q or R is complemented, then $\neg Q \dagger \neg R = \neg(Q ; R)$ and $\sim(Q \dagger R) = \sim Q ; \sim R$,*
2. *if Q and R are complemented, then $Q ; R$ and $Q \dagger R$ are complemented and $Q \dagger R = \neg(\neg Q ; \neg R)$ and $Q ; R = \neg(\neg Q \dagger \neg R)$.*

Duplex p-allegories II

It was shown in the context of relation algebras that univalent relations are closed under dual composition. The proof requires that Q is complemented and, implicitly, that R is homogeneous.

Duplex p-allegories II

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Theorem

Let $\mathcal{R}$ be a duplex p-allegory. Then if $Q : A \to B,\ R : B \to B$ are univalent, then $Q \overset{+}{;} R$ is univalent.

Example

Consider the following two sets $A = \{a\}$ and $B = \{a, b\}$ and the relations $Q : B \to A$ and $R : A \to B$ defined by

$$Q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Then both relations are univalent; in fact, these relations are even maps (univalent and total). We have

$$Q \,\dot{;}\, R = \begin{pmatrix} 1 \sqcup 1 & 1 \sqcup 0 \\ 1 \sqcup 1 & 1 \sqcup 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \mathbb{T}_{BB}.$$

i.e., $Q \,\dot{;}\, R$ is not univalent.

Duplex Heyting allegories I

Definition

A duplex Heyting allegory $\mathcal{R} = \langle \mathrm{Obj}_{\mathcal{R}}, \mathrm{Mor}_{\mathcal{R}}, ;, \mathbb{I}, /, \overset{+}{;}, \mathbb{D}, \curlywedge, \sqcap, \sqcup, \rightarrow, \rightsquigarrow, \perp\!\!\!\perp, \top\!\!\!\top, \smile \rangle$ is a structure consisting of two Heyting allegories, i.e., $\langle \mathrm{Obj}_{\mathcal{R}}, \mathrm{Mor}_{\mathcal{R}}, ;, \mathbb{I}, /, \sqcap, \sqcup, \rightarrow, \perp\!\!\!\perp, \smile \rangle$ and $\langle \mathrm{Obj}_{\mathcal{R}}, \mathrm{Mor}_{\mathcal{R}}, \overset{+}{;}, \mathbb{D}, \curlywedge, \sqcup, \sqcap, \rightsquigarrow, \top\!\!\!\top, \smile \rangle$ are Heyting allegories.

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Definition

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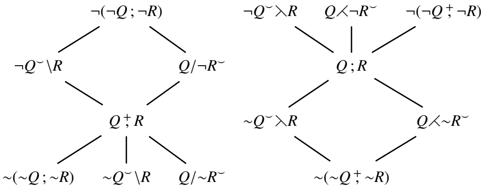
Lemma

Let $\mathcal{R}$ be a duplex Heyting allegory and $Q : A \to B$, $R : B \to C$. Then we have

1. $Q/{\sim}R^{\smile} \sqsubseteq Q \stackrel{+}{;} R \sqsubseteq Q/\neg R^{\smile}$,
2. ${\sim}Q^{\smile}\backslash R \sqsubseteq Q \stackrel{+}{;} R \sqsubseteq \neg Q^{\smile}\backslash R$,
3. $Q \curlywedge {\sim}R^{\smile} \sqsubseteq Q ; R \sqsubseteq Q \curlywedge \neg R^{\smile}$,
4. ${\sim}Q^{\smile} \curlyvee R \sqsubseteq Q ; R \sqsubseteq \neg Q^{\smile} \curlyvee R$.

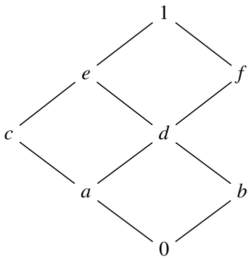
Hierarchy of residuals

The following diagram visualizes the order relationship between the construction mentioned above.



Example I

For this example we consider the following double Heyting algebra



Consider the following two relations and their pseudo- and quasi-complements

$$Q = \begin{pmatrix} e & 0 \\ e & d \end{pmatrix}, \; R = \begin{pmatrix} d & 0 \\ e & e \end{pmatrix},$$

$$\neg Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; {\sim}Q = \begin{pmatrix} f & 1 \\ f & 1 \end{pmatrix},$$

$$\neg R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; {\sim}R = \begin{pmatrix} 1 & 1 \\ f & f \end{pmatrix}$$

Example II

We compute

$$\neg(\neg Q;\neg R) = \neg\begin{pmatrix} (0\sqcap 0)\sqcup(1\sqcap 0) & (0\sqcap 1)\sqcup(1\sqcap 0) \\ (0\sqcap 0)\sqcup(0\sqcap 0) & (0\sqcap 1)\sqcup(0\sqcap 0)\end{pmatrix} = \neg\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix},$$

$$Q/\neg R^{\smile} = \begin{pmatrix} (0\to e)\sqcap(0\to 0) & (1\to e)\sqcap(0\to 0) \\ (0\to e)\sqcap(0\to d) & (1\to e)\sqcap(0\to d)\end{pmatrix} = \begin{pmatrix} 1\sqcap 1 & e\sqcap 1 \\ 1\sqcap 1 & e\sqcap 1\end{pmatrix} = \begin{pmatrix} 1 & e \\ 1 & e\end{pmatrix},$$

$$\neg Q^{\smile}\backslash R = \begin{pmatrix} (0\to d)\sqcap(1\to e) & (0\to 0)\sqcap(1\to e) \\ (0\to d)\sqcap(0\to e) & (0\to 0)\sqcap(0\to e)\end{pmatrix} = \begin{pmatrix} 1\sqcap e & 1\sqcap e \\ 1\sqcap 1 & 1\sqcap 1\end{pmatrix} = \begin{pmatrix} e & e \\ 1 & 1\end{pmatrix},$$

$$Q \mathbin{\dot{;}} R = \begin{pmatrix} (e\sqcup d)\sqcap(0\sqcup e) & (e\sqcup 0)\sqcap(0\sqcup e) \\ (e\sqcup d)\sqcap(d\sqcup e) & (e\sqcup 0)\sqcap(d\sqcup e)\end{pmatrix} = \begin{pmatrix} e\sqcap e & e\sqcap e \\ e\sqcap e & e\sqcap e\end{pmatrix} = \begin{pmatrix} e & e \\ e & e\end{pmatrix},$$

$$Q/{\sim}R^{\smile} = \begin{pmatrix} (1\to e)\sqcap(f\to 0) & (1\to e)\sqcap(f\to 0) \\ (1\to e)\sqcap(f\to d) & (1\to e)\sqcap(f\to d)\end{pmatrix} = \begin{pmatrix} e\sqcap 0 & e\sqcap 0 \\ e\sqcap e & e\sqcap e\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ e & e\end{pmatrix},$$

$${\sim}Q^{\smile}\backslash R = \begin{pmatrix} (f\to d)\sqcap(f\to e) & (f\to 0)\sqcap(f\to e) \\ (1\to d)\sqcap(1\to e) & (1\to 0)\sqcap(1\to e)\end{pmatrix} = \begin{pmatrix} e\sqcap e & 0\sqcap e \\ d\sqcap e & 0\sqcap e\end{pmatrix} = \begin{pmatrix} e & 0 \\ e & 0\end{pmatrix},$$

$${\sim}({\sim}Q;{\sim}R) = {\sim}\begin{pmatrix} (f\sqcap 1)\sqcup(1\sqcap f) & (f\sqcap 1)\sqcup(1\sqcap f) \\ (f\sqcap 1)\sqcup(1\sqcap f) & (f\sqcap 1)\sqcup(1\sqcap f)\end{pmatrix} = {\sim}\begin{pmatrix} f & f \\ f & f\end{pmatrix} = \begin{pmatrix} c & c \\ c & c\end{pmatrix}.$$

All of those relations are different. Even the meet of the relations above and the join of the relations below $Q \mathbin{\dot{;}} R$ do not lead to $Q \mathbin{\dot{;}} R$ as the following computations show

$$Q/\neg R^{\smile} \sqcap \neg Q^{\smile}\backslash R = \begin{pmatrix} 1 & e \\ 1 & e\end{pmatrix} \sqcap \begin{pmatrix} e & e \\ 1 & 1\end{pmatrix} = \begin{pmatrix} e & e \\ 1 & e\end{pmatrix},$$

$$Q/{\sim}R^{\smile} \sqcup {\sim}Q^{\smile}\backslash R \sqcup {\sim}({\sim}Q;{\sim}R) = \begin{pmatrix} 0 & 0 \\ e & e\end{pmatrix} \sqcup \begin{pmatrix} e & 0 \\ e & 0\end{pmatrix} \sqcup \begin{pmatrix} c & c \\ c & c\end{pmatrix} = \begin{pmatrix} e & c \\ e & e\end{pmatrix}.$$

Conclusion and Future Work

1. Proof the Archimedean property of the reals in the context of duplex allegories.
2. Consider arrow categories with an underlying duplex allegory.
3. . . .