

RIGHT-LINEAR STRUCTURES

$(\omega$ -)REGULAR LANGUAGES, FIXED POINTS AND CYCLIC PROOFS

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- 1 Systems for regular expressions
- 2 Inlining fixed points: algebras for automata
- 3 Cyclic proofs
- 4 Up to ω : alternating parity automata
- 5 Recovering axiomatic completeness results
- 6 Conclusions and further directions

Regular expressions over \mathcal{A} :

$$e, f, \dots ::= 0 \mid 1 \mid a \in \mathcal{A} \mid e + f \mid ef \mid e^*$$

Write $\mathcal{L}(e) \subseteq \mathcal{A}^*$ for the **regular language** computed by e .

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Definition (Kleene Algebra)

A **Kleene Algebra** (KA) is an **idempotent semiring** equipped with an operation $*$ s.t.:

$$\begin{array}{l|l} e^* = 1 + ee^* & e^* = 1 + e^*e \\ ef \leq f \implies e^*f \leq f & ef \leq e \implies ef^* \leq f \end{array}$$

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A **left-handed Kleene Algebra** (ℓ KA) is defined like KA but without second column.

NB: in a ℓ KA we have $e^* = \text{LFP}[X \mapsto 1 + eX]$.

EXAMPLES

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Algebra **Lang** of languages

The set of **languages** $\mathcal{P}(\mathcal{A}^*)$ where:

- 0 is \emptyset and 1 is $\{\varepsilon\}$.
- + is union and \cdot is concatenation.
- * is the usual Kleene * of a language.

Algebra **Rel** of relations

The set of **binary relations** $\mathcal{P}(\mathcal{A} \times \mathcal{A})$ where:

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Proposition (Folklore)

Lang, **Rel** and \mathcal{L} have the *same equational theory*.

Theorem ([Koz94, Kro90])

$$\mathcal{L}(e) \subseteq \mathcal{L}(f) \implies \text{KA} \models e \leq f$$

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- A celebrated but nontrivial result: Krob's proof is 137 pages!
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NB: This means KA axiomatises the equational theory of **Rel** too.

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In fact, via Krob's argument, we can obtain a stronger result:

Theorem (Left-handed completeness, [Bof95])

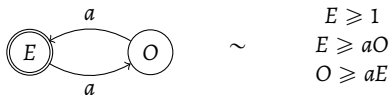
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Alternative (and shorter!) proofs are now known, [KS12, DDP18].

DIGGING DEEPER: NFAs AS RIGHT-LINEAR SYSTEMS

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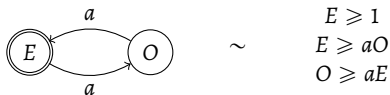
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$$\begin{aligned} E &\geq 1 \\ E &\geq aO \\ O &\geq aE \end{aligned}$$

$$\begin{aligned} \mathcal{L}(E) &= \{a^{2n}\}_{n \in \mathbb{N}} \\ \mathcal{L}(O) &= \{a^{2n+1}\}_{n \in \mathbb{N}} \end{aligned} \text{ are the } \textit{least solutions} \text{ in } \mathcal{L}.$$

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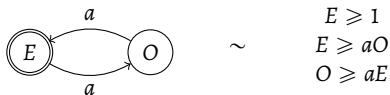
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Formalised Kleene Theorem

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Formalised Kleene Theorem

Every right-linear system has **least solutions** in ℓKA .

Motto: ℓKAs are *just* idempotent semirings with least solutions to NFAs.

The theory of idempotent semirings admits a **natural proof calculus**.

Sequents: $\Gamma \rightarrow e$, where Γ is a list of regular expressions (read $\prod \Gamma \leq e$)

Rules: *Lambek calculus* (equivalently, non-commutative IMALL)

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Possibilities for $*$

[Jip04] Induction rule:
$$\frac{\Gamma \rightarrow f \quad e, f \rightarrow f}{e^*, \Gamma \rightarrow f}$$

Not complete without cut

[Pal07] ω -rule:
$$\frac{\Gamma \rightarrow f \quad e, \Gamma \rightarrow f \quad e, e, \Gamma \rightarrow f \quad \dots}{e^*, \Gamma \rightarrow f}$$

Proofs are necessarily infinite

[DP17] 'Cyclic proofs' with unfoldings:
$$\frac{\Gamma \rightarrow f \quad e, e^*, \Gamma \rightarrow f}{e^*, \Gamma \rightarrow f}$$

Not regular complete without cut

The only finitary approach we have is via complex ‘hypersequents’, $\Gamma \rightarrow S$ where now S is a **set of lists** (read $\prod \Gamma \leq \sum_{\Delta \in S} \prod \Delta$).

Theorem ([DP17])

*There is a hypersequential calculus HKA **regularly cut-free complete** for \mathcal{L} .*

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Theorem ([DPI17])

There is a hypersequential calculus HKA *regularly cut-free complete* for \mathcal{L} .

This can be extended to ω -**regular** languages too!

Definition

$A \subseteq \mathcal{A}^\omega$ is ω -**regular** if $A = \bigcup_{i < n} B_i C_i^\omega$, where all $B_i, C_i \subseteq \mathcal{A}^+$ are regular.

Theorem ([HK22])

There is an extension of HKA *regularly complete* for ω -regular inclusions (without cut).

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RLA EXPRESSIONS: A NOTATION FOR NFAS

Right-Linear Algebra expressions (RLA expressions), e, f, \dots , are generated by:

$$e, f, \dots ::= 0 \mid 1 \mid X \mid ae \mid e + f \mid \mu Xe$$

NB: there is **no native product**.

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Language semantics:

- $\mathcal{L}(\mu X e(X)) := \text{LFP}[A \mapsto \mathcal{L}(e(A))]$

Knaster-Tarski Theorem $\implies \mathcal{L}(e)$ is well-defined.

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Example

We can solve the previous systems for E, O in two (equivalent) ways:

$$\begin{array}{ll} E = \mu X(1 + aaX) & O = \mu X(a + aaX) \\ O = aE & E = 1 + aO \end{array}$$

Right-linear expressions admit a natural **algebraic theory** analogous to ℓ KA:

Definition

A **right-linear algebra** (RLA) is a structure $\mathcal{L} = (L, 0, 1, +, \mathcal{A})$ where:

- $(L, 0, +)$ is a bounded join-semilattice.
- Each $a \in \mathcal{A}$ is a bounded semilattice **homomorphism**.
- Each right-linear system has **least solutions**.

NB: no axioms for 1.

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Example

Each ℓ KA is a RLA, but not vice-versa!

- The structures **Lang** and **Rel** form RLAs in the expected way.
- $\mathcal{P}(\mathcal{A}^\omega)$ forms an RLA in the expected way, but not a ℓ KA.

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Sequents: $e \rightarrow \Gamma$, where Γ is a set of expressions (read $e \leq \sum \Gamma$)

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$$\text{id} \frac{}{e \rightarrow e} \quad \text{h}_a \frac{e \rightarrow \Gamma}{ae \rightarrow a\Gamma} \quad \text{wk} \frac{e \rightarrow \Gamma}{e \rightarrow \Gamma, f}$$

Left logical rules:

$$\text{o-l} \frac{}{0 \rightarrow \Gamma} \quad \text{+l} \frac{e \rightarrow \Gamma \quad f \rightarrow \Gamma}{e + f \rightarrow \Gamma} \quad \mu\text{-l} \frac{e(\mu X e(X)) \rightarrow \Gamma}{\mu X e(X) \rightarrow \Gamma}$$

Right logical rules:

$$\text{o-r} \frac{e \rightarrow \Gamma}{e \rightarrow \Gamma, 0} \quad \text{+-r} \frac{e \rightarrow \Gamma, f_i}{e \rightarrow \Gamma, f_0 + f_1} \quad \mu\text{-r} \frac{e \rightarrow \Gamma, f(\mu X f(X))}{e \rightarrow \Gamma, \mu X f(X)}$$

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NB: the **fixed point rules** do *not* guarantee **leastness** of $\mu \dots$

Definition

- **Preproofs** are generated **coinductively** from the inference rules.
- A preproof is **cyclic/regular** if it has only **finitely many** distinct sub-preproofs.
- A **proof** is a preproof where each infinite branch has a 'good formula trace'.

Write CRLA for the class of cyclic $\widehat{\text{RLA}}$ proofs.

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Example

Write $a^* := \mu X(1 + aX)$. We can show $a^* = E + O$:

$$\begin{array}{c}
 \vdots \\
 \frac{\mu^{-l}, \mu^{-r}}{a^* \rightarrow O, E} \bullet \\
 \frac{\text{id} \quad \frac{h_a}{aa^* \rightarrow aO, aE}}{1 \rightarrow 1} \\
 \frac{+{-}l, +{-}r}{1 + aa^* \rightarrow 1 + aO, aE} \\
 \frac{\mu^{-l}, \mu^{-r}}{a^* \rightarrow E, O} \bullet \\
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SOME METALLOGICAL RESULTS

Theorem (Soundness)

$\text{CRLA} \vdash e \rightarrow f \implies \mathcal{L}(e) \subseteq \mathcal{L}(f)$.

Proof idea.

- For each $w \in \mathcal{L}(e)$ we take its **finite 'run'** along a cyclic proof.
- By induction on the run we show $w \in \mathcal{L}(f)$. □

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$\mathcal{L}(e) \subseteq \mathcal{L}(f) \implies \text{CRLA} \vdash e \rightarrow f$.

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- Define a bottom-up validity-preserving **proof search strategy**.
- Critical **loop-check** for weakenings.
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NB: This gives us an **effective algorithm** for proof search.

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We can reason about ω -regular languages via **dualising the syntax**.

RLL expressions:

$$e, f, \dots ::= X \mid ae \mid 0 \mid e + f \mid \mu X e \\ \mid \top \mid e \cap f \mid \nu X e$$

NB: we omit 1 to limit to strictly infinite words, for simplicity.

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Language semantics:

- $\mathcal{L}(\top) := \mathcal{A}^\omega$
- $\mathcal{L}(e \cap f) := \mathcal{L}(e) \cap \mathcal{L}(f)$.
- $\mathcal{L}(\nu X e(X)) := \text{GFP}[A \mapsto \mathcal{L}(e(A))]$

NB: Knaster-Tarski Theorem $\implies \mathcal{L}(e)$ is well-defined.

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RLL expressions are a **notation for alternating parity automata (APA)**.

This syntax can be seen as an *alternative* to the modal logic μLTL .

EXAMPLES

$\top : \{a, b\}^\omega$

a, b



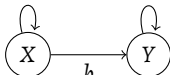
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$$\nu X(aX + bX)$$

f_a : "finitely many as"

a, b

b



1

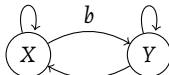
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$$\mu X(aX + bX + \nu Y(bY))$$

i_a : "infinitely many as"

a

b



0

a

1

$$\nu X \mu Y(aX + bY)$$

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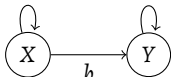


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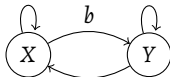
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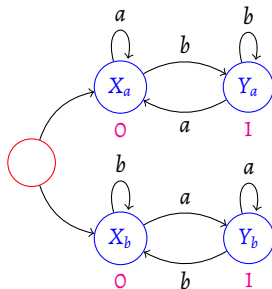
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$$\nu X \mu Y(aX + bY)$$

$$\nu X_a \mu Y_a(aX_a + bY_a)$$

\cap

$$\nu X_b \mu Y_b(aY_b + bX_b)$$



Key



: universal state



: existential state

n : priority

EXAMPLES

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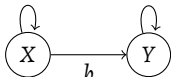


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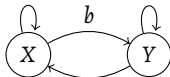
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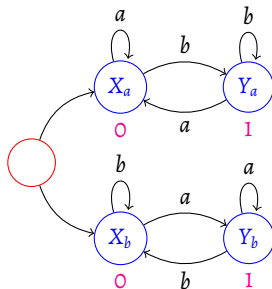
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$$\nu X_b \mu Y_b(aY_b + bX_b)$$



Key

: universal state

: existential state

n : priority

NB: equivalence between an expression and associated APA is **not immediate**...

SEGUE: THE EVALUATION GAME

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Positions: pairs (w, e) where $w \in \mathcal{A}^\omega$ and e an expression.

Players: *Eloise* (\exists) and *Abelard* (\forall).

Position	Player	Available moves
(aw, ae)	-	(w, e)
(aw, be) with $a \neq b$	\exists	
(w, \perp)	\exists	
(w, \top)	\forall	
$(w, e + f)$	\exists	$(w, e), (w, f)$
$(w, e \cap f)$	\forall	$(w, e), (w, f)$
$(w, \mu X e(X))$	-	$(w, e(\mu X e(X)))$
$(w, \nu X e(X))$	-	$(w, e(\nu X e(X)))$

\exists **wins** an infinite play if **smallest** infinitely occurring expression is a ν -expression.

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(aw, ae)	-	(w, e)
(aw, be) with $a \neq b$	\exists	
(w, \perp)	\exists	
(w, \top)	\forall	
$(w, e + f)$	\exists	$(w, e), (w, f)$
$(w, e \cap f)$	\forall	$(w, e), (w, f)$
$(w, \mu X e(X))$	-	$(w, e(\mu X e(X)))$
$(w, \nu X e(X))$	-	$(w, e(\nu X e(X)))$

\exists **wins** an infinite play if **smallest** infinitely occurring expression is a ν -expression.

By the now standard method of **signatures** (cf. [SE89]) we have:

Theorem (Adequacy)

$w \in \mathcal{L}(e) \iff \exists$ has a (positional) winning strategy from (w, e) .

\rightsquigarrow an RLL expression and its associated APA compute the **same ω -language**.

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 \top\text{-r} \frac{}{\rightarrow \top} \qquad \cap\text{-r} \frac{\Gamma \rightarrow \Delta, e \quad \Gamma \rightarrow \Delta, f}{\Gamma \rightarrow \Delta, e \cap f} \qquad \nu\text{-r} \frac{\Gamma \rightarrow \Delta, e(\nu X e(X))}{\Gamma \rightarrow \Delta, \nu X e(X)} \\
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 \text{p-l} \frac{}{ae, bf \rightarrow} a \neq b \qquad \text{h}_a \frac{\Gamma \rightarrow \Delta}{a\Gamma \rightarrow a\Delta} \Gamma \neq \emptyset \qquad \text{p-r} \frac{\{\rightarrow \Gamma_a\}_{a \in \mathcal{A}}}{\rightarrow \{a\Gamma_a\}_{a \in \mathcal{A}}}
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EXAMPLE

“Any ω -word over $\{a, b\}$ has finitely many a s or infinitely many a s”

$$\begin{array}{c}
 \vdots \\
 \frac{\vdots}{\vdots} \bullet \quad \frac{\mu-r, +r}{\vdots} \frac{\top \rightarrow af_a, bf_a, b^\omega, i'_a}{\top \rightarrow f_a, b^\omega, i'_a} \circ \\
 \frac{h_a}{a\top \rightarrow af_a, ai_a} \quad \frac{h_a}{b\top \rightarrow bf_a, bb^\omega, bi'_a} \\
 \frac{+l}{a\top + b\top \rightarrow af_a, bf_a, bb^\omega, ai_a, bi'_a} \\
 \frac{+r}{a\top + b\top \rightarrow af_a, bf_a, bb^\omega, ai_a + bi'_a} \\
 \frac{\nu-l, \nu-r, \mu-r}{\top \rightarrow af_a, bf_a, b^\omega, i'_a} \circ \\
 \frac{+r}{\top \rightarrow af_a + bf_a + b^\omega, i'_a} \bullet \\
 \frac{\mu-r, \nu-r}{\top \rightarrow f_a, i'_a} \bullet \\
 \frac{+r}{\top \rightarrow f_a + i'_a}
 \end{array}$$

Key

$$b^\omega := \nu Y(bY)$$

$$i'_a := \mu Y(ai_a + bY)$$

Correctness

$-\circ^\omega$: orange trace good

$-\bullet^\omega$: green trace good

$(-\bullet -\circ)^\omega$: green/blue trace good

Theorem (Soundness)

$\text{CRL} \vdash e \rightarrow f \implies \mathcal{L}(e) \subseteq \mathcal{L}(f)$.

NB: for metalogical analysis of CRL, we follow the now standard Niwinski-Walukiewicz machinery for finite state cyclic systems, cf. [NW96].

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Proof idea.

Let P be a CRL proof of $e \rightarrow f$.

- Suppose $w \in \mathcal{L}(e)$ and let σ be a \exists winning strategy from (w, e) .
- σ determines an infinite branch B_σ of P , following σ along the LHS.
- B_σ in turn induces a \exists winning strategy from (w, f) , by the RHS. □

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By **ω -regularity** of the correctness condition for CRLl:

Lemma (cf. Büchi-Landweber)

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$\mathcal{L}(e) \subseteq \mathcal{L}(f) \implies \text{CRLl} \vdash e \rightarrow f$ (for e, f guarded)

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Proof idea.

- Similar proof search strategy to CRLl, but guardedness \implies **fairness**.
- $\text{CRLl} \Vdash e \rightarrow f \implies$ there is a **'bad branch'** of proof search, by **determinacy**.
- By reduction to Evaluation Game, we extract a word $w \in \mathcal{L}(e) \setminus \mathcal{L}(f)$. □

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Definition

A **right-linear lattice** (RLL) is a RLL-structure $\mathfrak{L} = (L, 0, \top, +, \cap, \mu, \nu, \mathcal{A})$ s.t.:

- 1 $(L, 0, \top, +, \cap)$ is a bounded distributive lattice.
- 2 Each $a \in \mathcal{A}$ is a lower-bounded-lattice homomorphism.
- 3 The ranges of $a \in \mathcal{A}$ partition L .
- 4 $e(\mu X e(X)) \leq \mu X e(X)$ and $e(f) \leq f \implies \mu X e(X) \leq f$.
- 5 $\nu X e(X) \leq e(\nu X e(X))$ and $f \leq e(f) \implies f \leq \nu X e(X)$.
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Theorem (Soundness & Completeness of RLL)

$$\text{RLL} \models e \leq f \iff \mathcal{L}(e) \subseteq \mathcal{L}(f).$$

This can be proved by **reduction** to completeness of μLTL [Kai95].

NB: Duality of μ and ν is **necessary** for completeness.

It is not (easily) possible to adapt the same method for RLA. However, we can exploit cyclic completeness of $\widehat{\text{RLA}}$:

Theorem (Invariant extraction)

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Proof idea.

- Inspired by previous approaches for ℓKA [KS12, DDP18].
- Can compute **intersections** of languages via right-linear systems.
Morally similar to the product construction on NFAs.
- Appropriate **local properties** by analysis of cyclic proofs.
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Corollary (Completeness of RLA)

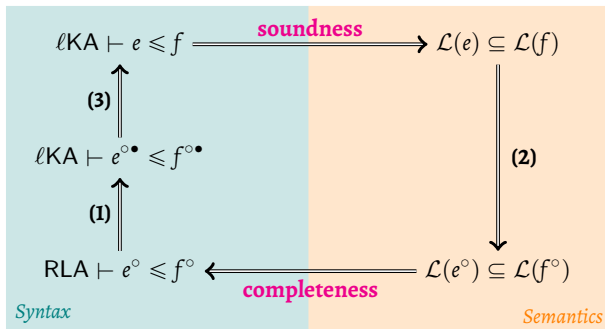
$$\mathcal{L}(e) \subseteq \mathcal{L}(f) \implies \text{RLA} \models e \leq f.$$

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BACK TO KLEENE ALGEBRA

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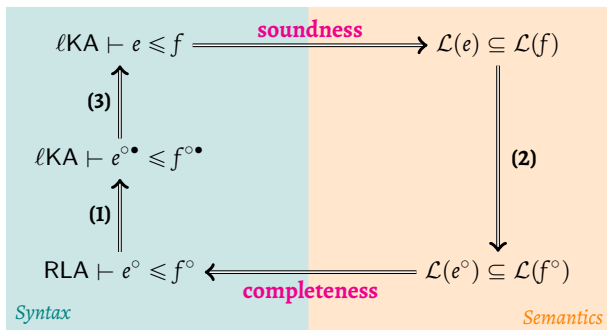
- 1 There is a **semantic interpretation** \circ from regular to RLL expressions.
- 2 There is a **proof interpretation** $\bullet : \text{RLA} \rightarrow \ell\text{KA}$.
- 3 \bullet and \circ are **ℓKA -compatible**: $\ell\text{KA} \vdash e^{\circ\bullet} = e$.



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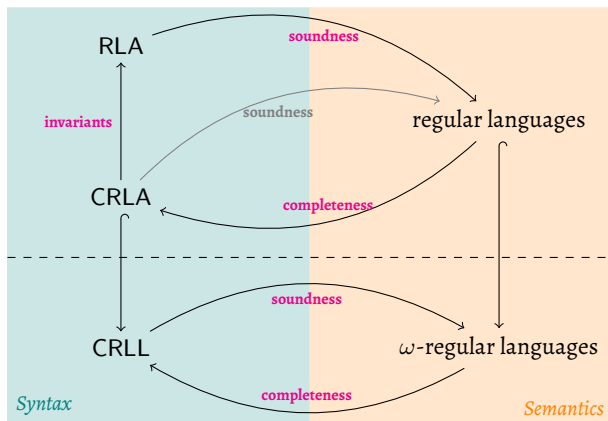
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Completeness of ℓKA is independent of multiplication, morally.

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SUMMARY



We can combine all the linguistic features we have seen so far:

$$e, f, \dots ::= 0 \mid X \mid a \mid e + f \mid ef \mid \mu X e \mid \nu X e$$

Such expressions compute just the ω -**context-free** languages.

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We obtain an extension $\mu\nu\ell$ HKA of HKA with:

Theorem (Assuming $\exists 0\#$)

$\mathcal{L}(e) \subseteq \mathcal{L}(f) \iff \mu\nu\ell$ HKA has a (not necessarily regular) proof of $e \rightarrow f$.

- Note that proofs are **inherently irregular!** Universality of CFLs is not r.e.
- We thus need **analytic determinacy** for completeness, which is beyond ZFC.

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Corollary

Completeness for an **infinitary axiomatisation** of CFLs (cf. [GH13]).

Perspectives

- Equational theories of automata via fixed point notations.
- Cyclic proofs a natural and powerful technique.
- Right-linear syntax much more amenable to proof theory.
- Completeness for regular languages is independent of multiplication!

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Future directions

- Further models e.g. **visibly pushdown** and **tree automata**.
- What about **computational interpretations** of proofs?
NB: beware **non-constructivity!**

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THANK YOU.



M. Boffa.

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