Combinatory Completeness in Structured Multicategories

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The Plan:

1. Combinatory Algebras and Combinatory Completeness
2. Faithful Cartesian Clubs and Structured Multicategories
3. Combinatory Completeness in Structured Multicategories
4. Bonus Results

1. Combinatory Algebras and Combinatory Completeness

An *applicative system* $(A, \bullet)$ consists of a set A together with a binary operation $\bullet : A \times A \to A$.

A convention: $\bullet$ is left-associative, infix, and usually omitted, as in

$$xyz = (xy)z = (x \bullet y) \bullet z = \bullet(\bullet(x, y), z)$$

Further examples:

$$xz(yz) = (x \bullet z) \bullet (y \bullet z) \qquad x(yzw)y = (x \bullet ((y \bullet z) \bullet w)) \bullet y$$

Say that an applicative system $(A, \bullet)$ has a(n):

- B combinator if $\exists \mathsf{B} \in A. \forall x, y, z \in A. \mathsf{B}xyz = x(yz)$
- C combinator if $\exists \mathsf{C} \in A. \forall x, y, z \in A. \mathsf{C}xyz = xzy$
- K combinator if $\exists \mathsf{K} \in A. \forall x, y \in A. \mathsf{K}xy = x$
- W combinator if $\exists \mathsf{W} \in A. \forall x, y \in A. \mathsf{W}xy = xyy$
- I combinator if $\exists \mathsf{I} \in A. \forall x \in A. \mathsf{I}x = x$

Then a BI-algebra is an applicative system with a B and I combinator, and so on.

A *combinatory algebra* is a BCKWI-algebra.

Some Examples:

Combinatory Logic (the free combinatory algebra)

Terms of the λ-calculus (open or closed) modulo $\equiv_\beta$

Various models of the λ-calculus (e.g., graph models)

There is a more structural chacterisation of combinatory algebras.

Fix an applicative system $(A, \bullet)$.

A *polynomial* in variables $x_1, \ldots, x_n$ is one of:

- a variable x_i where $1 \leq i \leq n$
- a combinator $a \in A$
- of the form $t \bullet s$ where t, s are polynomials in $x_1, \ldots, x_n$

E.g., if $a, b \in A$ then the following are polynomials in x, y, z:

$$a \bullet x \qquad a \qquad x \bullet (b \bullet z) \qquad a \bullet b \qquad y$$

A polymomial t in variables $x_1, \ldots, x_n$ is *computable* in case $\exists a \in A$ such that for all $b_1, \ldots, b_n \in A$ we have:

$$ab_1 \cdots b_n = t[b_1, \ldots, b_n / x_1, \ldots, x_n]$$

An applicative system is called *combinatory complete* in case all of its polynomials are computable.

Theorem (e.g., Curry & Feys 1958)

Let $(A, \bullet)$ *be an applicative system. Then* $(A, \bullet)$ *is combinatory complete if and only if it is a combinatory algebra (i.e., a* BCKWI-*algebra).*

A polynomial is *regular* in case it contains no constants.

For example the following are both polynomials in x_1, x_2, x_3

$$x_1(x_2 x_3) \qquad\qquad x_1 a$$

The one on the left is regular, but the one on the right is not.

To obtain a combinatory algebra it suffices to ask that all regular polynomials are computable.

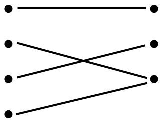
2. Faithful Cartesian Clubs and Structured Multicategories

The category $\mathbf{Fun}$ has:

Natural numbers as objects

Morphisms $\mathbf{a} : m \to n$ are functions $\mathbf{a} : \{1, \ldots, m\} \to \{1, \ldots, n\}$

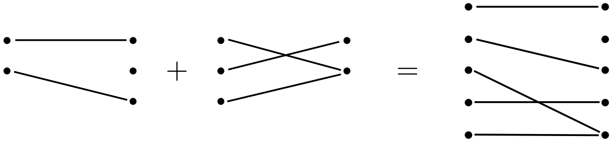
For example, this is a morphism $4 \to 3$ of $\mathbf{Fun}$:



$(\mathbf{Fun}, +, 0)$ is (cocartesian) strict monoidal

On objects, $+$ is addition of natural numbers

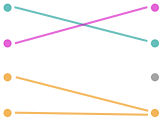
On morphisms, $+$ is defined as in:



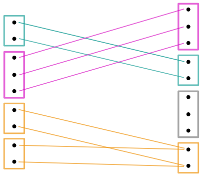
For each $\mathbf{a} : m \to n$ and $k_1, \ldots, k_n \in \mathbb{N}$ there is a *wreath product*:

$$\mathbf{a} \wr (k_1, \ldots, k_n) : \sum_{j=1}^{m} k_{\mathbf{a}(j)} \to \sum_{i=1}^{n} k_i$$

Definition by example. If $\mathbf{a} : 4 \to 4$ is:



Then $\mathbf{a} \wr (3, 2, 3, 2) : 9 \to 10$ is:



A *faithful cartesian club* is a wide subcategory of $\mathbf{Fun}$ that is closed under $+$ (from the monoidal structure) and $\wr$ (the wreath product).

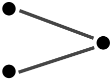
Club $\mathfrak{S}$	Consists of
Id	identities
Bij	bijections
Minj	monotone injections
Inj	injections
Srj	surjections
Fun	functions

Table: Some faithful cartesian clubs

Notably, the monotone surjections and monotone functions do not form faithful cartesian clubs.

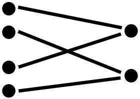
The following map $\mathbf{a} : 2 \to 1$ is a monotone surjection:

$$\mathbf{a} \quad \leftrightsquigarrow$$



But $\mathbf{a} \wr (2)$ is not monotone:

$$\mathbf{a} \wr (2) \quad \leftrightsquigarrow$$



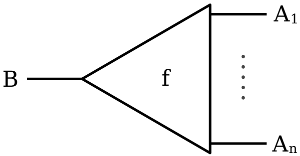
So these classes of function are not closed under wreath product.

A *multicategory* $\mathcal{M}$ has: (Part 1 of 2)

A set of *objects* $\mathcal{M}_0$

Sets of *morphisms* $\mathcal{M}(A_1, \ldots, A_n; B)$ for each $A_1, \ldots, A_n, B \in \mathcal{M}$

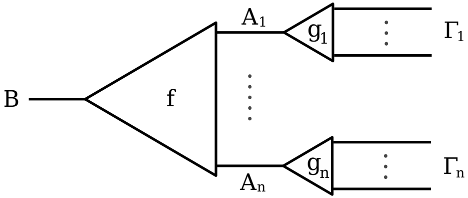
Identity morphisms $1_A \in \mathcal{M}(A; A)$ for each $A \in \mathcal{M}_0$



A *multicategory* $\mathcal{M}$ has: (Part 2 of 2)

For each $f \in \mathcal{M}(A_1, \ldots, A_n; B)$ and $(g_i \in \mathcal{M}(\Gamma_i; A_i))_{i \in \{1,\ldots,n\}}$

A *composite* $f \circ (g_1, \ldots, g_n) \in \mathcal{M}(\Gamma_1, \ldots, \Gamma_n; B)$

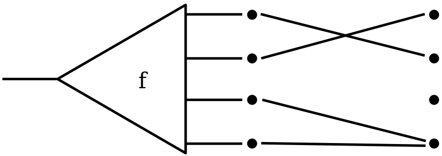


Satisfying sensible associativity and unitality axioms.

For $\mathfrak{S}$ a faithful cartesian club, an $\mathfrak{S}$-*multicategory* is a multicategory $\mathcal{M}$ equipped with an operation:

$$\mathcal{M}(A_{\mathbf{a}(1)}, \ldots, A_{\mathbf{a}(m)}; B) \xrightarrow{[-]\mathbf{a}} \mathcal{M}(A_1, \ldots, A_n; B)$$

for each $\mathbf{a} : m \to n$ of $\mathfrak{S}$, satisfying sensible axioms.

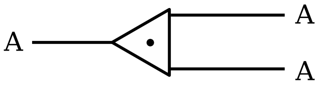


For example, there is a $\mathbf{Fun}$-multicategory Set where Set$(A_1, \ldots, A_n; B)$ is the set of functions $A_1 \times \cdots A_n \to B$.

3. Combinatory Completeness in Structured Multicategories

Fix a faithful cartesian club $\mathfrak{S}$ and an $\mathfrak{S}$-multicategory $\mathcal{M}$.

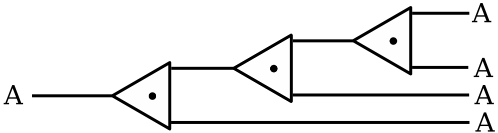
An *applicative system in* $\mathcal{M}$ is $(A, \bullet)$ where $\bullet \in \mathcal{M}(A, A; A)$.



We define *iterated application* $\bullet^n \in \mathcal{M}(A, A^n; A)$ for each $n \in \mathbb{N}$:

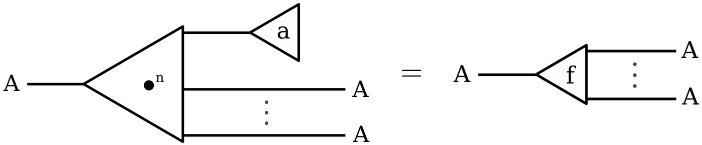
$$\bullet^0 = 1_A \qquad\qquad \bullet^{n+1} = \bullet \circ (\bullet^n, 1_A)$$

So that for example $\bullet^3 \in \mathcal{M}(A, A, A, A; A)$ is:



and $\bullet^1 = \bullet \in \mathcal{M}(A, A; A)$.

We say that $f \in \mathcal{M}(A^n; A)$ is *computable* in case there exists some $a \in \mathcal{M}(;A)$ such that $\bullet^n \circ (a, 1_A, \ldots, 1_A) = f$, as in:



All $a \in \mathcal{M}(;A)$ are computable as in $\bullet^0 \circ (a) = 1_A \circ (a) = a$.

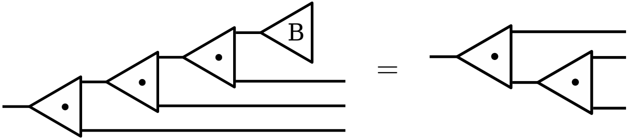
Define the *regular* $\mathfrak{S}$*-polynomials* over $(A, \bullet)$ to be the smallest sub-$\mathfrak{S}$-multicategory of $\mathcal{M}$ containing $\bullet \in \mathcal{M}(A, A; A)$.

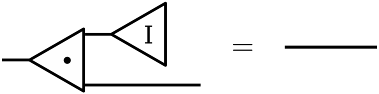
Say that $(A, \bullet)$ is *weakly* $\mathfrak{S}$*-combinatory complete* in case every $\mathfrak{S}$-polynomial over $(A, \bullet)$ is computable.

Define the $\mathfrak{S}$-*polynomials* over $(A, \bullet)$ to be the smallest sub-$\mathfrak{S}$-multicategory of $\mathcal{M}$ containing $\bullet \in \mathcal{M}(A, A; A)$ and all $a \in \mathcal{M}(; A)$.

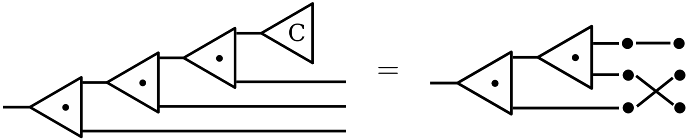
Say that $(A, \bullet)$ is $\mathfrak{S}$-*combinatory complete* in case every $\mathfrak{S}$-polynomial over $(A, \bullet)$ is computable.

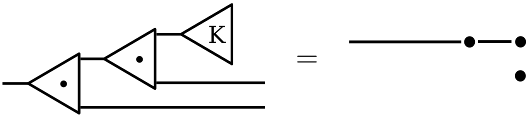
Combinators: (Part 1 of 2)

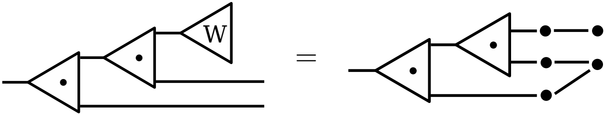




Combinators: (Part 2 of 2)







Theorem(s) about weak $\mathfrak{S}$-combinatory completeness:

Club $\mathfrak{S}$	Consists of	Characterises
Id	identities	BI-algebras
Bij	bijections	BCI-algebras
Minj	monotone injections	BKI-algebras
Inj	injections	BCKI-algebras
Srj	surjections	BCWI-algebras
Fun	functions	BCKWI-algebras

Table: Weak $\mathfrak{S}$-combinatory completeness results

For example, an aplicative system in a **Bij**-multicategory is weakly **Bij**-combinatory complete iff it is a BCI-algebra.

WARNING

From here we overtake the proceedings paper

(Preprint of journal version coming soon!)

What about (non-weak) $\mathfrak{S}$-combinatory completeness?

Definition (After Tomita)

We say that $(A, \bullet)$ is *flipped* in case for all $a \in \mathcal{M}$ there exists $h \in \mathcal{M}$ such that $\bullet \circ (h, 1_A) = \bullet \circ (1_A, a)$.

For example, any BCI-algebra is flipped.

Lemma

A weakly $\mathfrak{S}$-combinatory complete applicative system is flipped if and only if it is $\mathfrak{S}$-combinatory complete.

In particular, weak $\mathfrak{S}$-combinatory completeness and $\mathfrak{S}$-combinatory completeness coincide when $\mathbf{Bij} \subseteq \mathfrak{S}$.

4. Bonus Results

Combinatory completeness is categorical in nature:

Theorem

Let $\mathcal{M}$ be an $\mathfrak{S}$-multicategory, and $(A, \bullet)$ be an applicative system in $\mathcal{M}$. Then $(A, \bullet)$ is $\mathfrak{S}$-combinatory complete if and only if the $(A, \bullet)$-computable morphisms form a sub-$\mathfrak{S}$-multicategory of $\mathcal{M}$.

Extensionality coresponds to closed multicategory structure.

Definition

Say that $(A, \bullet)$ is *multi-extensional* in case for all $n \in \mathbb{N}$ and $a, b \in \mathcal{M}(; A)$, if $\bullet^n \circ (a, 1_A^n) = \bullet^n \circ (b, 1_A^n)$ then $a = b$.

Theorem

An $\mathfrak{S}$-combinatory complete applicative system $(A, \bullet)$ in $\mathcal{M}$ is multi-extensional if and only if the sub-multicatgory of $(A, \bullet)$-computable morphisms is closed with $[A, A] = A$ and $\mathrm{ev}_{A,A} = \bullet$.

However, this kind of extensionality is nonstandard.

The usual notion of extensionality is:

Definition

Let $\mathcal{M}$ be a multicategory, and let $(A, \bullet)$ be an applicative system in $\mathcal{M}$. Say that $(A, \bullet)$ is *extensional* in case for all $a, b \in \mathcal{M}(; A)$, if $\bullet \circ (a, 1_A) = \bullet \circ (b, 1_A)$ then $a = b$.

This is often enough:

Definition

Say that a multicategory $\mathcal{M}$ is *well-pointed* in case for all $f, g \in \mathcal{M}(A_1, \ldots, A_n; B)$, if $f \circ (x_1, \ldots, x_n) = g \circ (x_1, \ldots, x_n)$ for all $(x_i \in \mathcal{M}(; A_i))_{i=1}^n$ then $f = g$.

In particular, Set is point-separated. We have:

Lemma

If $\mathcal{M}$ is well-pointed, then any $(A, \bullet)$ in $\mathcal{M}$ is extensional if and only if it is multi-extensional.

End of Talk

Thanks for Listening!