Contractions of quasi relation algebras and applications to representability

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Outline

Definition

An FL-algebra is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ such that

- $\langle A, \wedge, \vee\rangle$ is a lattice,
- $\langle A, \cdot, 1\rangle$ is a monoid,
- $a \cdot b \leqslant c \iff b \leqslant a\backslash c \iff a \leqslant c/b$, and
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An InFL-algebra is cyclic if $-a = \sim a$ for all $a \in A$.

Alternative presentation of InFL-algebras

Lemma (Galatos and Jipsen 2013)

An involutive FL-algebra (InFL-algebra) is term-equivalent to an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \sim, -, 1\rangle$ *such that* $\langle A, \wedge, \vee\rangle$ *is a lattice,* $\langle A, \cdot, 1\rangle$ *is a monoid, and for all* $a, b, c \in A$*, we have*

$$a \cdot b \leqslant c \quad \iff \quad a \leqslant -(b \cdot {\sim}c) \quad \iff \quad b \leqslant {\sim}(-c \cdot a)\,.$$

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Residuals can be expressed in terms of $\cdot$ and the linear negations:

$$c/b = -(b \cdot {\sim}c) \quad \text{and} \quad a\backslash c = {\sim}(-c \cdot a).$$

Definition

A De Morgan InFL-algebra is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \sim, -, \neg, 1\rangle$ such that $\langle A, \wedge, \vee, \cdot, \sim, -, 1\rangle$ is an InFL-algebra and the following hold for all $a, b \in A$:

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A quasi relation algebra (qRA) is a De Morgan InFL-algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \sim, -, \neg, 1\rangle$ such that the following holds for all $a, b \in A$:

$$\text{(Dp)} \qquad \neg(a \cdot b) = \sim(-\neg b \cdot -\neg a)\,.$$

September 2020

Decidability of quasi relation algebras

Theorem (Galatos, J. 2013)

If $\mathcal{V} = \{\mathbf{A} \in \mathsf{InFL} \mid \mathbf{A} \models \mathcal{E}\}$ for $\mathcal{E}$ a self-dual set of identities and $\mathcal{V}$ is equationally decidable then $\mathcal{V}' = \{\mathbf{A} \in \mathsf{qRA} \mid \mathbf{A} \models \mathcal{E}\}$ is also decidable.

Using results of [Holland, McCleary 1979], [Yetter 1990], [Wille 2005], [Kozak 2011], [Galatos, J. (Res. Frames) 2013]:

Corollary

The equational theories of **qRA**, *cyclic* **qRA**, *cyclic distributive* **qRA**, *commutative* **qRA** *and the variety of qRAs that have ℓ-group reducts $(= \mathbf{A} \in \mathsf{qRA} \mid A \models x \cdot {\sim}x = 1)$ are decidable.*

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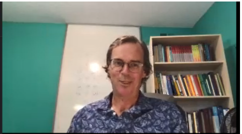
qRA, cyclic qRA and commutative qRA have FMP.

Problem 2: Do (cyclic) **distributive** qRA have the FMP?
Problem 3: Define and investigate **representable qRA**.



qRAs

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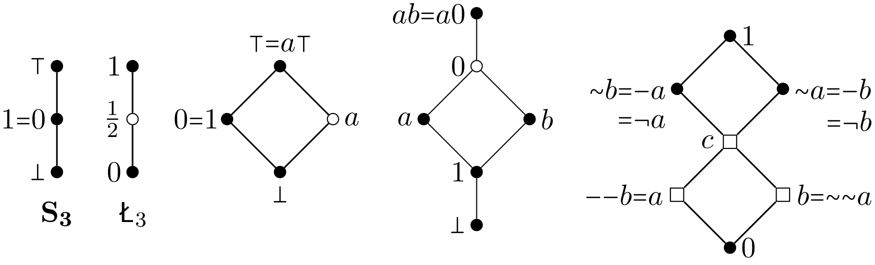
A distributive quasi relation algebra (DqRA) is a qRA such that the underlying lattice $\langle A, \wedge, \vee\rangle$ is distributive.

Examples of qRAs



Idempotents = black nodes, non-idempotents = empty nodes.
Circles = central elements, squares=non-central.

Full list of DqRAs up to size 8 by C., Jipsen, Robinson: DInFL1.pdf

Examples include all relation algebras, MV-algebras and Sugihara monoids.

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Let $\mathbf{A}$ be a qRA and $p \in A$. Define $pAp := \{pap \mid a \in A\}$.

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Lemma

Let $\mathbf{A}$ be a qRA and p a positive idempotent. Then $\langle pAp, \wedge, \vee \rangle$ is a sublattice of $\langle A, \wedge, \vee \rangle$ and $\langle pAp, \cdot \rangle$ is a subsemigroup of $\langle A, \cdot \rangle$.

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For $p \neq 1$, the set pAp cannot form a submonoid of $\langle A, \cdot, 1\rangle$ since $1 \notin pAp$.

Lemma

Let $\mathbf{A}$ be a qRA, p a positive symmetric idempotent of $\mathbf{A}$, and $b \in pAp$. Then

- $\sim b \in pAp$,
- $-b \in pAp$ *and*
- $\neg b \in pAp$.

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Theorem

Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \sim, -, \neg, 1 \rangle$ be a qRA and p a positive symmetric idempotent of $\mathbf{A}$. Then the algebra $p\mathbf{A}p = \langle pAp, \wedge, \vee, \cdot, \sim, -, \neg, p \rangle$ is a qRA.

Contractions of quasi relation algebras

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NB: if $\mathbf{A}$ is distributive, then so is $p\mathbf{A}p$.

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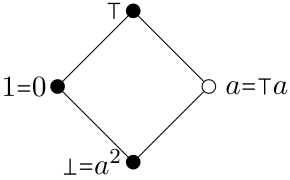
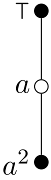
Compare with equivalence elements for RAs (Jónsson & Tarski 1952).

Examples of contractions of a qRA



$\mathbf{A}$



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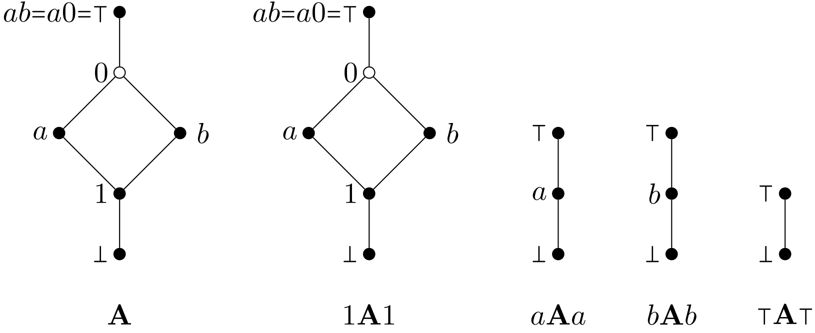

The algebras $p\mathbf{A}p$ for $\mathbf{A}$ and $p \in \{1, a, b, \top\}$.

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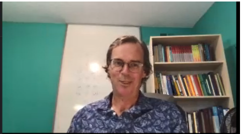

Some familiar operations on relations

Let X be a set, E an equivalence relation on X, and assume $R, S \subseteq E \subseteq X \times X$. We will use the following binary relations:

- $\varnothing$
- $\mathrm{id}_X = \{(x,x) \mid x \in X\}$
- $X \times X$
- $R\,;S = \{(x,y) \mid \exists z((x,z) \in R \text{ and } (z,y) \in S)\}$
- $R^{\smile} = \{(y,x) \mid (x,y) \in R\}$
- $E \backslash R = R^c = \{(x,y) \in E \mid (x,y) \notin R\}$
- $R \cup S$
- $R \cap S$

Constructing DqRAs of binary relations

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Hence $\langle \mathsf{Up}(\mathbf{E}), ;, \leqslant \rangle$ is a monoid.

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- If $R \in \mathsf{Up}(\mathbf{E})$, then $\alpha ; \beta ; R^{c} ; \beta \in \mathsf{Up}(\mathbf{E})$.

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$$\mathbf{Dq}(\mathbf{E}) = \langle \mathsf{Up}\left(\mathbf{E}\right), \cap, \cup, ;, \sim, -, \neg, \leqslant \rangle$$

is a distributive quasi relation algebra. If α *is the identity, then* $\mathbf{Dq}(\mathbf{E})$ *is a cyclic distributive quasi relation algebra.*

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The algebra $\mathbf{Dq}(\mathbf{E})$ is called an equivalence distributive quasi relation algebra.

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The algebra $\mathbf{Dq}(\mathbf{E})$ is called an equivalence distributive quasi relation algebra. If $E = X \times X$ it is called a full distributive quasi relation algebra.

Constructing DqRAs of binary relations

For $R \in \mathsf{Up}(\mathbf{E})$, define

$$\sim R = R^{c\smile} ; \alpha \qquad -R = \alpha ; R^{c\smile} \qquad \neg R = \alpha ; \beta ; R^{c} ; \beta$$

Theorem (C., Robinson 2025)

Let $\mathbf{X} = \langle X, \leqslant \rangle$ be a poset and E an equivalence relation on X such that $\leqslant \subseteq E$. Let $\alpha : X \to X$ be an order automorphism and $\beta : X \to X$ a self-inverse dual order automorphism of $\mathbf{X}$ such that $\alpha, \beta \subseteq E$ and $\beta = \alpha ; \beta ; \alpha$. Then

$$\mathbf{Dq}(\mathbf{E}) = \langle \mathsf{Up}(\mathbf{E}), \cap, \cup, ;, \sim, -, \neg, \leqslant \rangle$$

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It can be shown that

$$\mathbb{IP}(\mathsf{FDqRA}) = \mathbb{I}(\mathsf{EDqRA}).$$

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We say that a DqRA $\mathbf{A}$ is finitely representable if the poset $\mathbf{X} = \langle X, \leqslant \rangle$ used in the representation of $\mathbf{A}$ is finite.

Constructing $\mathbf{S}_2$

$\mathbf{S}_2$ is representable over $\mathbf{X} = \langle X, \leqslant \rangle$ with

- $X = \{x\}$,
- $\leqslant = \mathrm{id}_X = \{(x, x)\}$,
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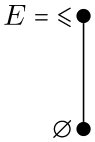

$$\mathbf{S}_2 \cong \mathbf{Dq}(\mathbf{E})$$

Constructing $\mathbf{S}_3$

$\mathbf{S}_3$ is representable over $\mathbf{X} = \langle X, \leqslant \rangle$ with

- $X = \{x, y\}$,
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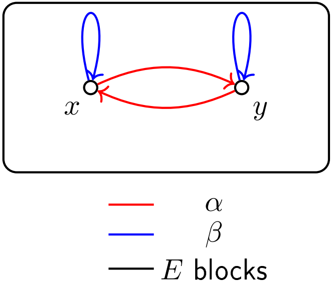

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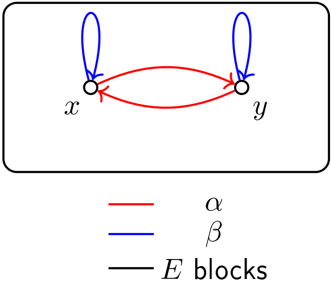
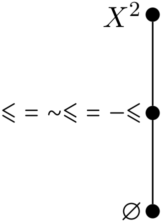


$$\mathbf{S}_3 \hookrightarrow \mathcal{Q}(\mathbf{E})$$



Example of a bigger representable DqRA

$\mathbf{A}$ is representable over $\mathbf{X} = \langle X, \leqslant \rangle$ with

- $X = \{w, x, y, z\}$,
- $\leqslant = \mathrm{id}_X = \{(w,w),(x,x),(y,y),(z,z)\}$,
- $E = X \times X$,
- $\alpha = \{(w,x),(x,w),(y,z),(z,y)\}$, and
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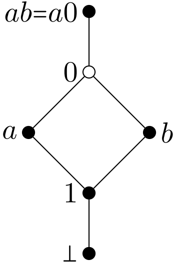

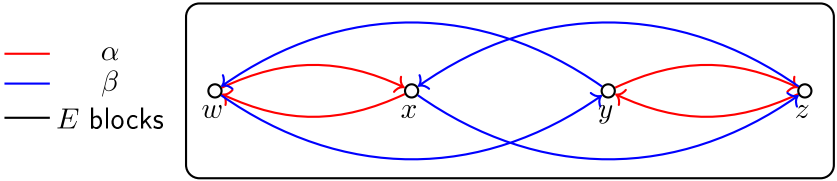

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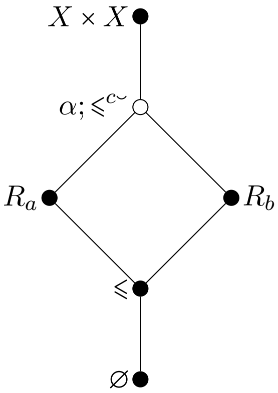

$$R_a = \{(w,w),(x,x),(y,y),(z,z),(w,y),(y,w),(x,z),(z,x)\}$$
$$R_b = \{(w,w),(x,x),(y,y),(z,z),(w,z),(z,w),(x,y),(y,x)\}$$

Representability of contractions of DqRAs

Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \sim, -, \neg, 1 \rangle$ be a representable DqRA and p a positive symmetric idempotent of $\mathbf{A}$.

Representability of contractions of DqRAs

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Then there exists

- a poset $\mathbf{X} = \langle X, \leqslant\rangle$,
- an equivalence relation $E \subseteq X \times X$ with $\leqslant \subseteq E$,
- an order automorphism $\alpha : X \to X$ such that $\alpha \subseteq E$,
- a self-inverse dual order automorphism $\beta : X \to X$ such that $\beta \subseteq E$ and $\alpha \,;\, \beta \,;\, \alpha = \beta$, and
- an embedding $\varphi : \mathbf{A} \hookrightarrow \langle \mathsf{Up}(\mathbf{E}), \cap, \cup, ;, \sim, -, \neg, \leqslant\rangle$.

The relation $\varphi(p)$ is a preorder on X such that $\leqslant \subseteq \varphi(p)$.

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Next define $E_{p\mathbf{A}p} \subseteq X/{\equiv} \times X/{\equiv}$ by

$$([x], [y]) \in E_{p\mathbf{A}p} \quad \text{iff} \quad (x, y) \in E.$$

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$E_{p\mathbf{A}p}$ is an equivalence relation on $X/{\equiv}$ such that $\leqslant_{X/\equiv} \subseteq E_{p\mathbf{A}p}$.

Representability of contractions of DqRAs

Now define $\alpha_{p\mathbf{A}p} : X/{\equiv} \to X/{\equiv}$ and $\beta_{p\mathbf{A}p} : X/{\equiv} \to X/{\equiv}$ by setting, for all $[x] \in X/{\equiv}$,

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Lemma

For all $x, y \in X$, we have $(x, y) \in \varphi(p)$ iff $(\alpha(x), \alpha(y)) \in \varphi(p)$ iff $(\beta(y), \beta(x)) \in \varphi(p)$.

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Lemma

1. *The map $\alpha_{p\mathbf{A}p} : X/{\equiv} \to X/{\equiv}$ is an order automorphism with $\alpha_{p\mathbf{A}p} \subseteq E_{p\mathbf{A}p}$.*
2. *The map $\beta_{p\mathbf{A}p} : X/{\equiv} \to X/{\equiv}$ is a self-inverse dual order automorphism with $\beta_{p\mathbf{A}p} \subseteq E_{p\mathbf{A}p}$.*
3. $\alpha_{p\mathbf{A}p} \,;\, \beta_{p\mathbf{A}p} \,;\, \alpha_{p\mathbf{A}p} = \beta_{p\mathbf{A}p}$

Theorem

The algebra $\mathbf{Dq}(\mathbf{E}_{p\mathbf{A}p}) = \left\langle \mathsf{Up}\left(\mathbf{E}_{p\mathbf{A}p}\right), \cap, \cup, ;, \sim, -, \neg, \leqslant_{X/\equiv} \right\rangle$ *is a DqRA.*

Representability of contractions of DqRAs

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Theorem

Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \sim, -, \neg, 1 \rangle$ *be a (finitely) representable DqRA, and let* p *be a positive symmetric idempotent of* $\mathbf{A}$*. Then* $p\mathbf{A}p$ *is (finitely) representable.*

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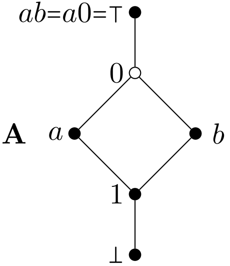
Proof.

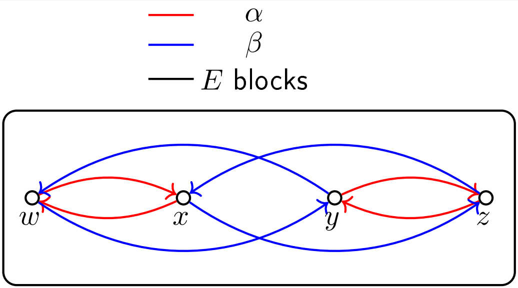
The map $\psi : pAp \to \mathsf{Up}\left(\mathbf{E}_{p\mathbf{A}p}\right)$ defined by

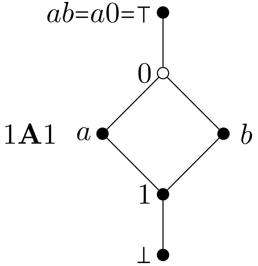
$$\psi(a) := \{([x],[y]) \mid (x,y) \in \varphi(a)\}$$

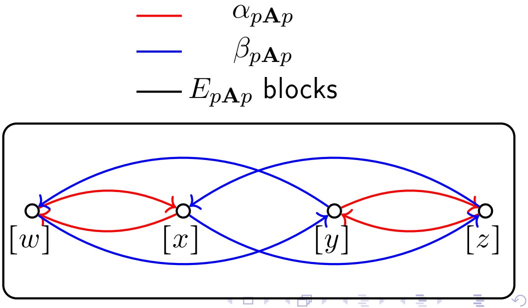
is an embedding from $p\mathbf{A}p$ into $\mathbf{Dq}(\mathbf{E}_{p\mathbf{A}p})$. □

Examples of the construction









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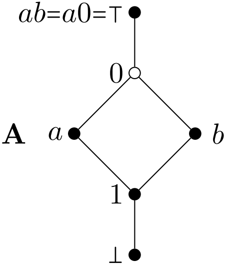
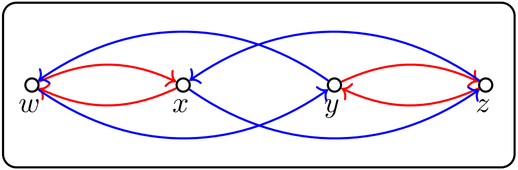


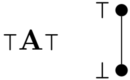
$\varphi(\top) = X \times X$

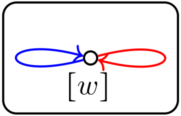
α

β

E blocks



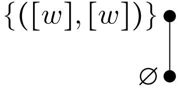




$\alpha_{\top\mathbf{A}\top}$

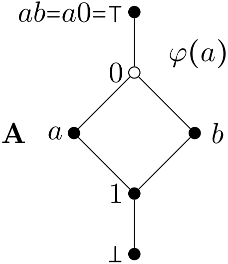
$\beta_{\top\mathbf{A}\top}$

$E_{\top\mathbf{A}\top}$ blocks



$\top\mathbf{A}\top \hookrightarrow \mathbf{Dq}(\mathbf{E}_{\top\mathbf{A}\top})$

Examples of the construction

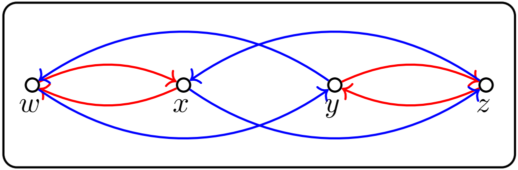


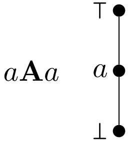
$$\varphi(a) = \leqslant \cup \{(w,y),(y,w),(x,z),(z,x)\}$$

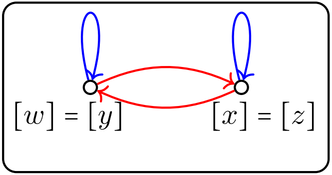
α

β

E blocks







$\alpha_{a\mathbf{A}a}$

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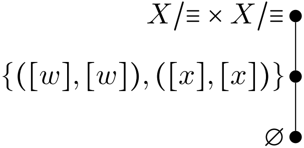
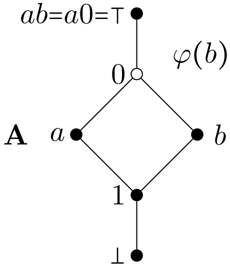
$E_{a\mathbf{A}a}$ blocks

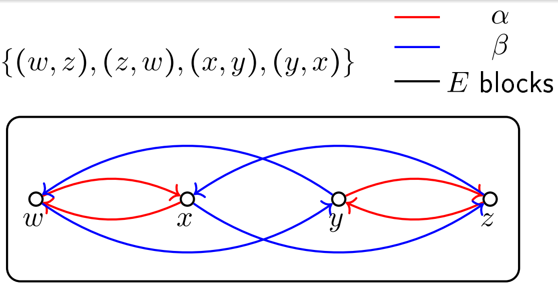


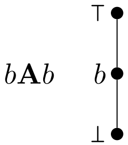
$$a\mathbf{A}a \hookrightarrow \mathbf{Dq}(\mathbf{E}_{a\mathbf{A}a})$$

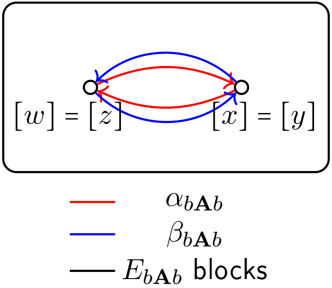
Examples of the construction

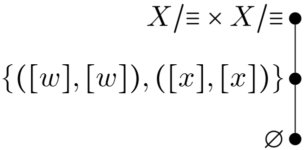


$\varphi(b) = \leqslant \cup \{(w,z),(z,w),(x,y),(y,x)\}$









$b\mathbf{A}b \hookrightarrow \mathbf{Dq}(\mathbf{E}_{b\mathbf{A}b})$

DqRAs that are not finitely representable

Earlier result:

Theorem (C., Robinson 2025)

Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \sim, -, \neg, 1 \rangle$ *be a DqRA. If there exists* $a \in A$ *such that* $-1 < a < 1$ *and* $a^2 \leqslant -1$*, then* $\mathbf{A}$ *is not finitely representable.*

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Application of contractions:

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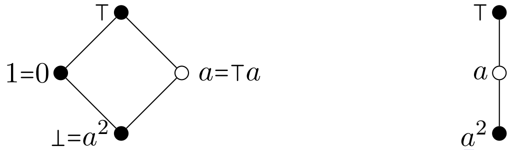
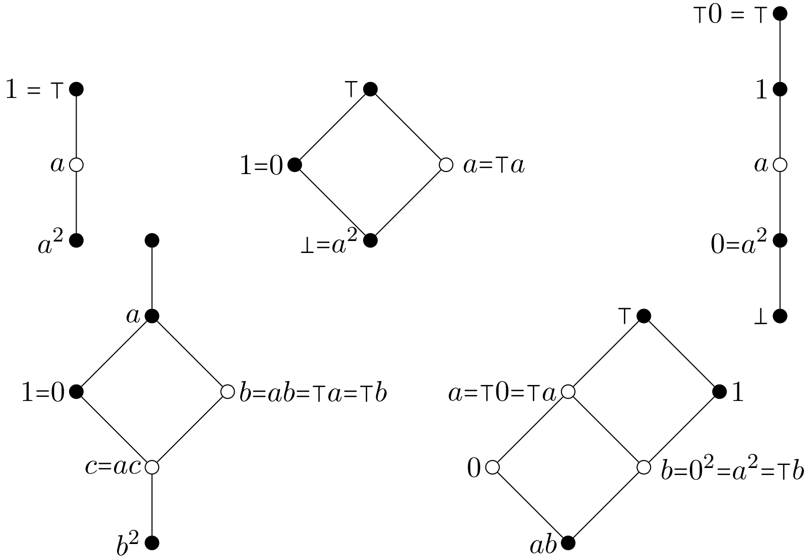


Examples of DqRAs that are not finitely representable



Future work

- Finding small non-representable DqRAs and then applying the contrapositive of the main result of this talk to obtain larger non-representable DqRAs.
- We can find larger representable DqRAs if we can prove: if the contraction $p\mathbf{A}p$ of a DqRA $\mathbf{A}$ with a positive symmetric idempotent p is representable, then $\mathbf{A}$ is representable.

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