

Szymczak Category and Leray Functor for Linear Relations

Mateusz Przybylski
Faculty of Mathematics and Computer Science
Jagiellonian University in Kraków

Based on joint work with B. Furmanek, F. Łanecki and J. Wiseman

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- 1 Dynamical motivation.
 - Continuous and discrete Conley index
 - Shift equivalence
 - Normal functors
 - Vietoris Theorem
- 2 Description for the category of finite length modules and linear relations.

Conley index [1] - setting

X - locally compact metric space

A continuous map $\varphi: \mathbb{R} \times X \rightarrow X$ is a *flow* provided

$$\begin{aligned}\varphi(0, x) &= x, \\ \varphi(t, \varphi(s, x)) &= \varphi(t + s, x).\end{aligned}$$

$S \subset X$ is an *invariant set* for φ if

$$\varphi(\mathbb{R}, S) := \bigcup_{t \in \mathbb{R}} \varphi(t, S) = S.$$

A compact set $N \subset X$ is an *isolating neighborhood* if

$$\text{Inv}(N, \varphi) := \{x \in N \mid \varphi(\mathbb{R}, x) \subset N\} \subset \text{int } N.$$

S is an *isolated invariant set* if $S = \text{Inv}(N)$ for some isolating neighborhood N

Conley index - setting

$\varphi_\lambda: \mathbb{R} \times X \rightarrow X, \lambda \in [-1, 1]$ - a continuous family of dynamical systems

Proposition ([1])

Let N be an isolating neighborhood for φ_0 . Then, for sufficiently small $\delta > 0$, set N is an isolating neighborhood for all φ_λ , where $|\lambda| < \delta$.

Definition ([1])

*Let $N \subset X$ be a compact set and $S_\lambda = \text{Inv}(N, \varphi_\lambda)$. Two isolated invariant sets S_{λ_0} and S_{λ_1} are **related by continuation** if N is an isolating neighborhood for all $\varphi_\lambda, \lambda \in [\lambda_0, \lambda_1]$.*

Conley index - expectations

- 1 If $\text{Inv}(N, \varphi) \subset \text{int } N$, $\text{Inv}(N', \varphi) \subset \text{int } N'$ and $\text{Inv}(N, \varphi) = \text{Inv}(N', \varphi)$, then the Conley index of N is the same as the index of N' .
- 2 If the Conley index of N is not trivial, then $\text{Inv } N \neq \emptyset$.
- 3 If $\text{Inv}(N, \varphi_\lambda) \subset \text{int } N$ for $\lambda \in [0, 1]$, then the Conley index of N under φ_0 is the same as the Conley index of N under φ_1 .

To avoid some technical difficulties the definition of the index works in the context of isolating neighborhoods, but we are interested in the structure and properties of invariant sets.

Pointed spaces

(Y, y_0) - a pointed topological space

(N, L) - a pair of spaces with $L \subset N$

$$N/L := (N \setminus L) \cup [L],$$

where $[L]$ denotes the equivalence class of points in L in the equivalence relation $x \sim y \Leftrightarrow x = y$ or $x, y \in L$.

$(N/L, [L])$ - pointed space (in short N/L)

Homotopy type of pointed space

$(X, x_0), (Y, y_0)$ - pointed spaces

$f, g: (X, x_0) \rightarrow (Y, y_0)$ - continuous functions with $f(x_0) = g(x_0) = y_0$

f is *homotopic* to g (denote $f \sim g$) if there exists a continuous function $F: X \times [0, 1] \rightarrow Y$ such that

$$F(x, 0) = f(x), \quad F(x, 1) = g(x), \quad F(x_0, s) = y_0 \text{ for } s \in [0, 1].$$

(X, x_0) and (Y, y_0) are *homotopic* (denote $(X, x_0) \sim (Y, y_0)$) if there exist $f: (X, x_0) \rightarrow (Y, y_0)$ and $g: (Y, y_0) \rightarrow (X, x_0)$ such that

$$f \circ g \sim \text{id}_Y \quad \text{and} \quad g \circ f \sim \text{id}_X.$$

Definition ([1])

Let S be an isolated invariant set. A pair of compact sets (N, L) where $L \subset N$ is called an *index pair* for S if:

- $S = \text{Inv}(\text{cl}(N \setminus L))$ and $N \setminus L$ is a neighborhood of S .
- If $x \in L$ and $\varphi([0, t], x) \subset N$, then $\varphi([0, t], x) \subset L$.
- If $x \in N$ and $t_1 > 0$ such that $\varphi(t_1, x) \notin N$, then there exists $t_0 \in [0, t_1]$ for which $\varphi([0, t_0], x) \subset N$ and $\varphi(t_0, x) \in L$.

Theorem

Given an isolated invariant set S , there exists an index pair.

Definition ([1])

The *homotopy Conley index* of S is

$$h(S) = h(S, \varphi) \sim (N/L, [L]).$$

Theorem ([1])

Let (N, L) and (N', L') be index pairs for an isolated invariant set S .
Then

$$(N/L, [L]) \sim (N'/L', [L']).$$

Definition ([1])

The *homological Conley index* of S is

$$CH_*(S) := H_*(N/L, [L]) \approx H_*(N, L).$$

Properties of the index

Theorem (Continuation property [1])

Let S_{λ_0} and S_{λ_1} be isolated invariant sets that are related by continuation. Then,

$$CH_*(S_{\lambda_0}) \approx CH_*(S_{\lambda_1}).$$

Theorem (Ważewski property [1])

Let N be an isolating neighborhood and assume that $CH_(\text{Inv } N) \not\approx 0$. Then, $\text{Inv } N \neq \emptyset$.*

Discrete dynamical systems

X - locally compact metric space

$f: X \rightarrow X$ - a continuous map

$N \subset X$. An *invariant set* of N is defined by

$$\text{Inv}(N, f) := \{x \in N \mid f^n(x) \in N \text{ for all } n \in \mathbb{Z}\}.$$

A compact set N is called an *isolating neighborhood* if

$$\text{Inv}(N, f) \subset \text{int } N.$$

A set S is an *isolated invariant set* if there exists an isolating neighborhood N such that $S = \text{Inv}(N, f)$.

Definition ([4])

A pair (N, L) of compact subsets of X is called an *index pair* for an isolated invariant set S if $L \subset N$ and

- $\text{cl}(N \setminus L)$ is an isolating neighborhood for S ,
- $f(L) \cap N \subset L$ (positive invariance),
- $f(N \setminus L) \subset N$ (exit set).

Theorem ([4])

For any neighborhood V of an isolated invariant set S there exists an index pair (N, L) for S such that $N \setminus L \subset V$.

Index map

For an index pair (N, L) , the *index map* is the map $f_{N,L}: N/L \rightarrow N/L$ defined by

$$f_{N,L}([x]) := \begin{cases} [f(x)] & \text{if } x, f(x) \in N \setminus L, \\ [L] & \text{otherwise.} \end{cases}$$

Proposition ([4])

For any index pair (N, L) the index map $f_{N,L}$ is continuous.

Shift equivalence [2]

\mathcal{C} - a category,

$A, B \in \mathcal{C}$ - objects,

$\alpha: A \rightarrow A, \beta: B \rightarrow B$ - endomorphisms,

(A, α) and (B, β) are *shift equivalent* (denoted $\alpha \sim \beta$) if there exist $m \in \mathbb{N}, r: A \rightarrow B$ and $s: B \rightarrow A$ such that

$$r \circ \alpha = \beta \circ r, \quad s \circ \beta = \alpha \circ s, \quad r \circ s = \beta^m, \quad s \circ r = \alpha^m.$$

Discrete Conley index

Definition ([2])

The *discrete Conley index* of S is defined as the shift equivalence class of $f_{N,L}$.

Theorem ([2])

If (N, L) and (N', L') are index pairs for an isolated invariant set S , then the index maps $f_{N,L}$ and $f_{N',L'}$ are shift equivalent.

Category of endomorphisms [4,5]

\mathcal{E} - a category,

$\text{END}(\mathcal{E})$ - a *category of endomorphisms* of \mathcal{E} :

(1) objects:

$$\text{Obj}(\text{END}(\mathcal{E})) := \{(E, e) \mid E \in \text{Obj}(\mathcal{E}) \text{ and } e \in \text{Mor}_{\mathcal{E}}(E, E)\};$$

(2) morphisms:

$$\text{Mor}_{\text{END}(\mathcal{E})}((E, e), (F, f)) := \{\varphi \in \text{Mor}_{\mathcal{E}}(E, F) \mid \varphi \circ e = f \circ \varphi\}.$$

Szymczak category [5]

\mathcal{E} - a category,

$\text{SZYM}(\mathcal{E})$ - a *Szymczak category* associated with \mathcal{E} :

(1) objects:

$$\text{Obj}(\text{SZYM}(\mathcal{E})) := \text{Obj}(\text{END}(\mathcal{E}));$$

(2) morphisms:

$$(E, e), (E', e') \in \text{Obj}(\text{SZYM}(\mathcal{E})),$$

$$\text{for } (\varphi, m), (\varphi', m') \in \text{Mor}_{\text{END}(\mathcal{E})}((E, e), (E', e')) \times \mathbb{N}$$

$$(\varphi, m) \equiv (\varphi', m') \Leftrightarrow \exists k \in \mathbb{N} \quad \varphi \circ e^{m'+k} = \varphi' \circ e^{m+k}.$$

(2) morphisms:

$$\text{Mor}_{\text{SZYM}(\mathcal{E})}((E, e), (E', e')) := \text{Mor}_{\text{END}(\mathcal{E})}((E, e), (E', e')) \times \mathbb{N} / \equiv .$$

(a) composition:

$$[\varphi, m]: (E, e) \rightarrow (E', e'), [\varphi', m']: (E', e') \rightarrow (E'', e'')$$

$$[\varphi', m'] \circ [\varphi, m] := [\varphi' \circ \varphi, m + m'] .$$

(b) the identity:

$$[\text{id}_E, 0]: (E, e) \rightarrow (E, e).$$

The Szymczak functor [5]

Define a functor $\text{SZYM}: \text{END}(\mathcal{E}) \rightarrow \text{SZYM}(\mathcal{E})$ such that:

(1) for $(E, e) \in \text{Obj}(\text{END}(\mathcal{E}))$

$$\text{SZYM}((E, e)) := (E, e),$$

(2) for $\varphi \in \text{Mor}_{\text{END}(\mathcal{E})}((E, e), (F, f))$

$$\text{SZYM}(\varphi) := [\varphi, 0].$$

Normal functor [4,5,6]

\mathcal{C}, \mathcal{E} - categories,

$F: \text{END}(\mathcal{E}) \rightarrow \mathcal{C}$ - a functor.

Definition

A functor F is called a *normal functor* if and only if $F(e)$ is an isomorphism in \mathcal{C} for any endomorphism $e \in \text{Mor}_{\text{END}(\mathcal{E})}((E, e), (E, e))$.

The Szymczak functor is normal

Every morphism $[e, 0]: (E, e) \rightarrow (E, e)$ in $\text{SZYM}(\mathcal{E})$ has an inverse given by $[\text{id}_E, 1]$. Indeed,

$$[e, 0] \circ [\text{id}_E, 1] = [e, 1] = [\text{id}_E, 0].$$

Thus, $\text{SZYM}(e) = [e, 0]$ is invertible in $\text{SZYM}(\mathcal{E})$.

Theorem ([5])

Szymczak functor is a normal functor.

Theorem ([5])

For every normal functor $F: \text{END}(\mathcal{E}) \rightarrow \mathcal{C}$ there exists a unique functor $F': \text{SZYM}(\mathcal{E}) \rightarrow \mathcal{C}$ such that the diagram

$$\begin{array}{ccc} \text{END}(\mathcal{E}) & \xrightarrow{F} & \mathcal{C} \\ \text{SZYM} \downarrow & \nearrow F' & \\ \text{SZYM}(\mathcal{E}) & & \end{array}$$

commutes.

Different point of views

- Szymczak functor and category,
- shift equivalences:

isomorphism classes in the Szymczak category are equivalence classes of shift equivalence

- localization of END category:

the Szymczak category can be seen as a localization of the $\text{END}(\mathcal{E})$ category with respect to the class of morphisms $e \in \text{END}((E, e), (E, e))$.

The Leray functor [4] - step one

$\text{END}(\mathcal{E})$ - category of endomorphisms of graded modules,
 $(E, e) \in \text{END}(\mathcal{E})$

The *generalized kernel* of e is defined as

$$\text{gker}(e) := \bigcup \{e^{-n}(0) \mid n \in \mathbb{N}\}.$$

We have induced monomorphism

$$e' : E / \text{gker}(e) \ni [x] \mapsto [e(x)] \in E / \text{gker}(e),$$

because $e(\text{gker}(e)) \subset \text{gker}(e)$.

$$LM(E, e) := (E / \text{gker}(e), e')$$

The Leray functor - step one

$\varphi: (E, e) \rightarrow (F, f)$ - a morphism

We have induced mapping

$$\varphi': E/\text{gker}(e) \rightarrow F/\text{gker}(f),$$

because $\varphi(\text{gker}(e)) \subset \text{gker}(f)$.

$$LM(\varphi) := \varphi'.$$

LM goes from $\text{END}(\mathcal{E})$ to a full subcategory of $\text{END}(\mathcal{E})$ consisting of graded modules equipped with monomorphism.

The Leray functor - step two

$(E, e) \in \text{END}(\mathcal{E})$ - an object equipped with monomorphism

The *generalized image* of e is defined as

$$\text{gim}(e) := \bigcap \{e^n(E) \mid n \in \mathbb{N}\}.$$

The contraction

$$e'' : \text{gim}(e) \ni x \mapsto e(x) \in \text{gim}(e)$$

is a monomorphism and an epimorphism.

$$LI(E, e) := (\text{gim}(e), e'')$$

The Leray functor - step two

$\varphi: (E, e) \rightarrow (F, f)$ - a morphism of objects equipped with monomorphisms

We have contraction

$$\varphi'': \text{gim}(e) \ni x \mapsto \varphi(x) \in \text{gim}(f).$$

$$LI(\varphi) := \varphi''.$$

LI goes from a full subcategory of $\text{END}(\mathcal{E})$ consisting of graded modules with monomorphisms to a full subcategory of $\text{END}(\mathcal{E})$ consisting of graded modules with isomorphisms

The Leray functor [4]

$$LM(E, \mathbf{e}) := (E / \text{gker}(\mathbf{e}), \mathbf{e}')$$

$$LM(\varphi) := \varphi'$$

$$LI(E, \mathbf{e}) := (\text{gim}(\mathbf{e}), \mathbf{e}')$$

$$LI(\varphi) := \varphi''$$

The *Leray functor* is defined as

$$L := LI \circ LM$$

Multivalued extensions for Conley theory

$f: X \rightarrow Y$ - a continuous map

$F \subset X \times Y$ - a relation

$p: F \rightarrow X$ and $q: F \rightarrow Y$ - canonical projections.

Assume

- $p(F) = X$,
- $f(x) \in q(p^{-1}(x))$ for all $x \in X$,
- $p^{-1}(x)$ is acyclic for all $x \in X$.

Theorem ([7])

Let F and X be compact metric spaces. Let $p: F \rightarrow X$ be a surjective continuous function. If $p^{-1}(x)$ is acyclic for every $x \in X$, then $p_: H_*(F) \rightarrow H_*(X)$ is an isomorphism.*

Then, $F_* = q_* \circ (p_*)^{-1}$.

Linear relations

A, B - R -modules

A subset $\alpha \subset A \oplus B$ is called a *linear relation* if it is a submodule of $A \oplus B$ (denoted $\alpha: A \multimap B$).

Let $\alpha: A \multimap A$ be a linear relation. The *generalized kernel* of α is defined as

$$\text{gker } \alpha := \left\langle \bigcup_{I \in \mathbb{Z}} \alpha^I(\mathbf{0}) \right\rangle,$$

and the *generalized image* of α is defined as

$$\text{gim } \alpha := \bigcap_{I \in \mathbb{Z}} \alpha^I(A).$$

The Leray functor for $\text{REL}(R)$ [3]

$\text{REL}(R)$ - the category whose objects are R -modules and whose morphisms are linear relations

$(A, \alpha) \in \text{ENDREL}(R)$

$$\iota_A: \text{gim } \alpha \ni a \mapsto a \in A$$

$$\pi_A: A \ni a \mapsto [a] \in A/\text{gker } \alpha$$

$$\text{LE}(A, \alpha) := (\text{gim } \alpha, \iota_A^{-1} \circ \alpha \circ \iota_A),$$

$$\text{LM}(A, \alpha) := (A/\text{gker } \alpha, \pi_A \circ \alpha \circ \pi_A^{-1}).$$

The Leray functor for $\text{REL}(R)$

$(B, \beta) \in \text{ENDREL}(R)$ and $\varphi: (A, \alpha) \multimap (B, \beta)$

$$\text{LE}(\varphi) := (\iota_B^{-1} \circ \varphi \circ \iota_A)$$

$$\text{LM}(\varphi) := (\pi_B \circ \varphi \circ \pi_A^{-1})$$

The *Leray functor* is defined as

$$L := \text{LE} \circ \text{LM}.$$

Theorem ([3])

The Leray functor $L: \text{ENDREL}(R) \rightarrow \text{ENDREL}(R)$ is a normal functor.

Classifying strength of functors

\mathcal{C} - a category, $A, B \in \mathcal{C}$,

$$A \overset{\mathcal{C}}{\simeq} B \Leftrightarrow A \text{ and } B \text{ are isomorphic in } \mathcal{C}$$

\mathcal{D} - a category, $F: \mathcal{C} \rightarrow \mathcal{D}$ - a functor

$$A \overset{F}{\simeq} B \text{ if } F(A) \overset{\mathcal{D}}{\simeq} F(B)$$

\mathcal{D}' - a category, $F': \mathcal{C} \rightarrow \mathcal{D}'$ - a functor

Definition

*We say that F is finer than F' , if $A \overset{F}{\simeq} B$ implies $A \overset{F'}{\simeq} B$ for any $A, B \in \mathcal{C}$. If F is finer than F' and F' is finer than F , then both functors have equal **classifying strength**.*

Classifying strength of Leray and Szymczak functors

Szymczak functor is finer than Leray functor but the converse fails:

$$R = \mathbb{Z}$$

$$(\cdot 2): \mathbb{Z} \ni x \multimap 2 \cdot x \in \mathbb{Z} \quad \text{and} \quad (\cdot 0): \mathbb{Z} \ni x \multimap 0 \in \mathbb{Z}$$

$$L(\mathbb{Z}, (\cdot 2)) = L(\mathbb{Z}, (\cdot 0)) = (0, \text{id}_0), \text{ hence } (\mathbb{Z}, (\cdot 2)) \stackrel{L}{\simeq} (\mathbb{Z}, (\cdot 0)).$$

Assume $[\varphi, n]: (\mathbb{Z}, (\cdot 2)) \multimap (\mathbb{Z}, (\cdot 0))$ and $[\psi, m]: (\mathbb{Z}, (\cdot 0)) \multimap (\mathbb{Z}, (\cdot 2))$ are inverse morphisms in category $\text{SZYMR}(\mathbb{Z})$, then $\varphi(\mathbb{Z}) = \{0\}$.

On the other hand, for $z \neq 0$ we have

$0 \notin (\cdot 2)^{k+m+n}(z) = (\psi \circ \varphi \circ (\cdot 2)^k)(z) = \psi(0)$. By linearity, $0\psi 0$, a contradiction.

Modules of finite length

Definition

Let M be an R -module. We say that M is *Noetherian* and *Artinian*, if every increasing and decreasing sequence of its submodules stabilizes, respectively.

A module is of *finite length* if it is both Noetherian and Artinian. We define $\text{FREL}(R)$ to be the full subcategory of $\text{REL}(R)$, whose objects are R -modules of finite length.

Case of \mathbb{Z}_p fields

p - a prime number

$\top := \mathbb{Z}_p \oplus \mathbb{Z}_p$, $\perp := \{(0, 0)\}$ and $(\cdot\lambda): \mathbb{Z}_p \ni x \mapsto \lambda \cdot x \in \mathbb{Z}_p$ for $\lambda \in \mathbb{Z}_p$

Proposition ([3])

There are exactly $p + 3$ linear relations on \mathbb{Z}_p , that is $\top, \perp, (\cdot 0)^{-1}$ and $(\cdot\lambda)$ for $\lambda \in \mathbb{Z}_p$.

Proposition ([3])

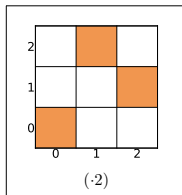
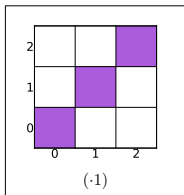
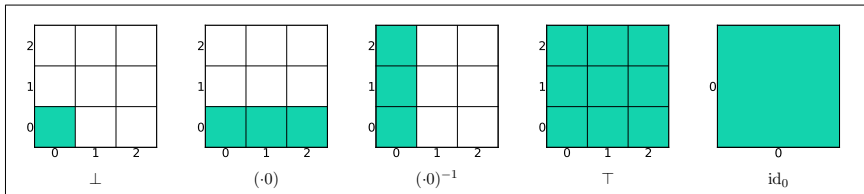
Let $\lambda, \mu \in \mathbb{Z}_p$. If $(\mathbb{Z}_p, (\cdot\lambda)) \stackrel{\text{SZYM}}{\simeq} (\mathbb{Z}_p, (\cdot\mu))$, then $\lambda = \mu$.

\mathcal{C} - the full subcategory of FREL(\mathbb{Z}_p) with only two objects, which are the zero space and the \mathbb{Z}_p field

Proposition ([3])

The objects (\mathbb{Z}_p, \top) , (\mathbb{Z}_p, \perp) , $(\mathbb{Z}_p, (\cdot 0))$, $(\mathbb{Z}_p, (\cdot 0)^{-1})$ and $(0, \text{id}_0)$ are isomorphic in SZYMC

Example for $p = 3$



General case

Lemma ([3])

Let $(A, \alpha) \in \text{ENDFREL}(R)$. Then $(A, \alpha) \stackrel{\text{SZYM}}{\simeq} \text{LE}(A, \alpha)$.

Lemma ([3])

Let $(A, \alpha) \in \text{ENDFREL}(R)$. Then $(A, \alpha) \stackrel{\text{SZYM}}{\simeq} \text{LM}(A, \alpha)$.

Theorem ([3])

$(A, \alpha) \stackrel{\text{SZYM}}{\simeq} L(A, \alpha)$ for every $(A, \alpha) \in \text{ENDFREL}(R)$.

Main results for linear relations

Theorem ([3])

Let $(A, \alpha), (B, \beta) \in \text{ENDFREL}(R)$. Then $(A, \alpha) \stackrel{\text{SZYM}}{\simeq} (B, \beta)$ if and only if $(A, \alpha) \stackrel{L}{\simeq} (B, \beta)$.

Theorem ([3])

Let $(A, \alpha), (B, \beta) \in \text{ENDFREL}(R)$ be such that α and β are bijections. If $(A, \alpha) \stackrel{\text{SZYM}}{\simeq} (B, \beta)$ then $(A, \alpha) \simeq (B, \beta)$.

Corollary ([3])

For any $(A, \alpha) \in \text{ENDFREL}(R)$ we have $\text{LE} \circ \text{LM}(A, \alpha) \simeq \text{LM} \circ \text{LE}(A, \alpha)$.

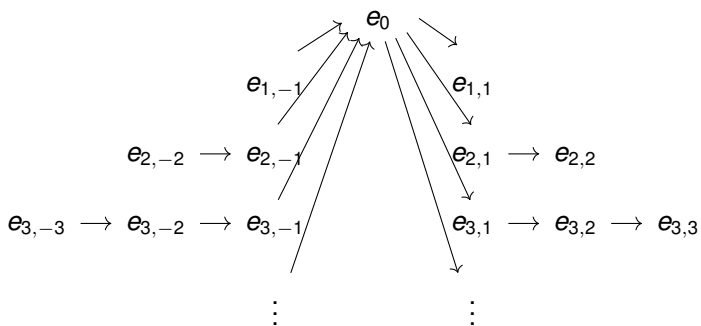
Counterexample [3]

Order of composition of LE and LM in the Leray functor matters.

For $k \in \mathbb{N}_+$ let $E_k := \{e_{k,s} : s = -k, -k+1, \dots, k-1, k\}$, where $e_{k,0} = e_{k',0} =: e_0$ for $k, k' \in \mathbb{N}_+$.

Let β_k be a relation on E_k defined as $e_{k,s} \beta_k e_{k,s+1}$ for $-k \leq s < k$.

Thus $\beta := \bigsqcup_{k \in \mathbb{N}_+} \beta_k$ is a relation (not yet linear) on $E := \bigsqcup_{k \in \mathbb{N}_+} E_k$.



Counterexample

A - a free \mathbb{Z}_2 -vector space with basis E

$$\alpha := \langle \beta \rangle \subseteq A \oplus A$$

It can be proved that $\text{gim } \alpha \not\subseteq \alpha(\text{gim } \alpha)$ and $\text{gim } \alpha \not\subseteq \alpha^{-1}(\text{gim } \alpha)$.

$\text{LE}(A, \alpha) = (\mathbb{Z}_2, \perp)$ because $\text{gim } \alpha = \{0, e_0\}$. Thus,
 $\text{LM} \circ \text{LE}(A, \alpha) = (\mathbb{Z}_2, \perp)$.

Also $\text{LM} \circ \text{LE}(\alpha) = \perp$ (α is treated as an endomorphism of (A, α) in $\text{ENDREL}(\mathbb{Z}_2)$).

However, $\perp: (\mathbb{Z}_2, \perp) \dashv\vdash (\mathbb{Z}_2, \perp)$ is not an isomorphism, but $\text{LE} \circ \text{LM}(\alpha)$ must be an isomorphism.

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Thank you for your attention.