Games on Higher Dimensional Automata

Emily Clement [1,2] Léo Henry [3], Benjamin Monmege [3]

[1]CNRS, UMR 7030, F-93430, Villetaneuse, France
[2]Université Sorbonne Paris Nord, LIPN, F-93430, Villetaneuse, France
[3]Aix-Marseille Univ, CNRS, LIS, Marseille, France

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Games on Automata

- Player 

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- Environment 

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- Player ●
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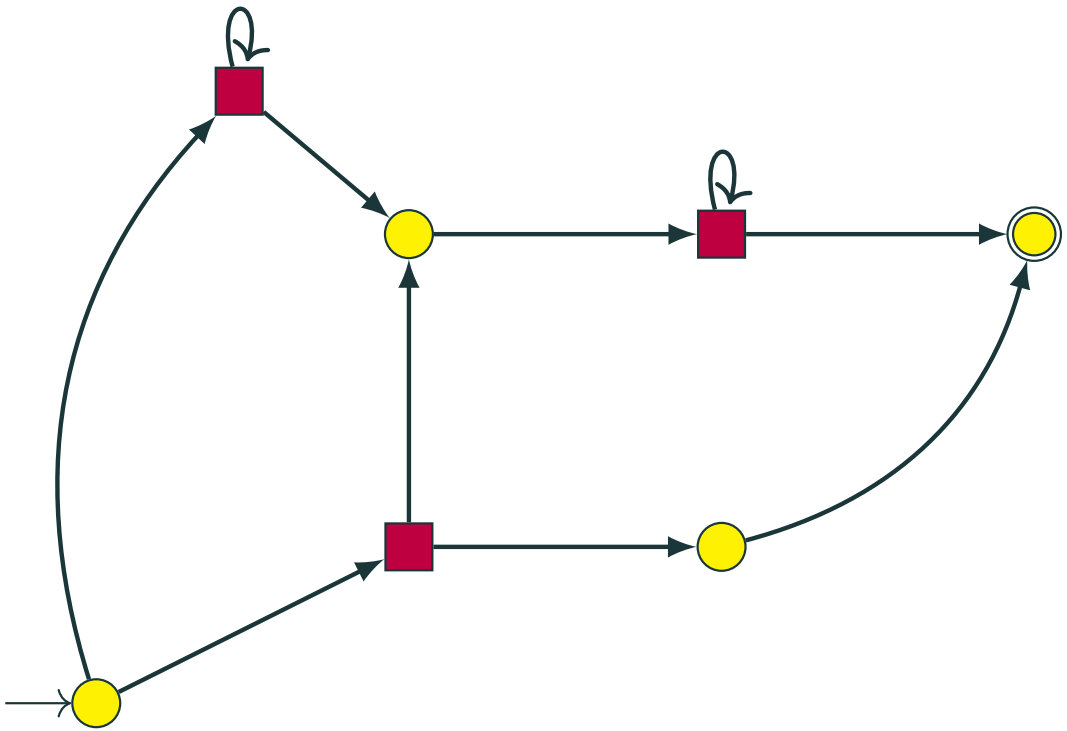

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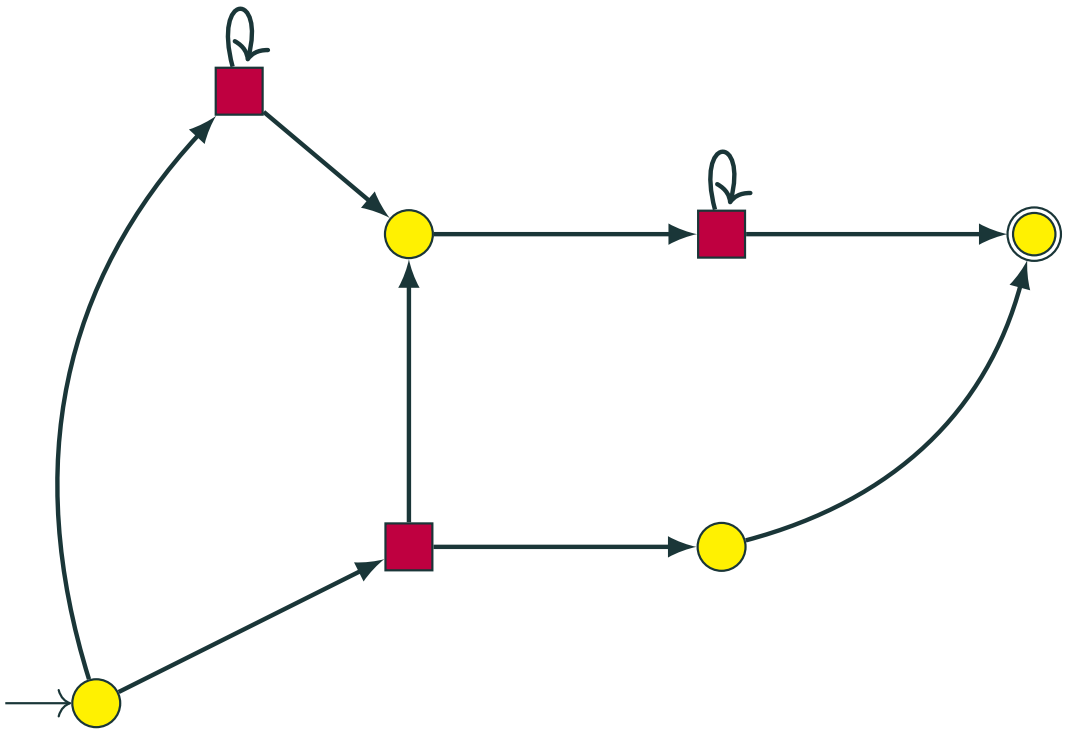

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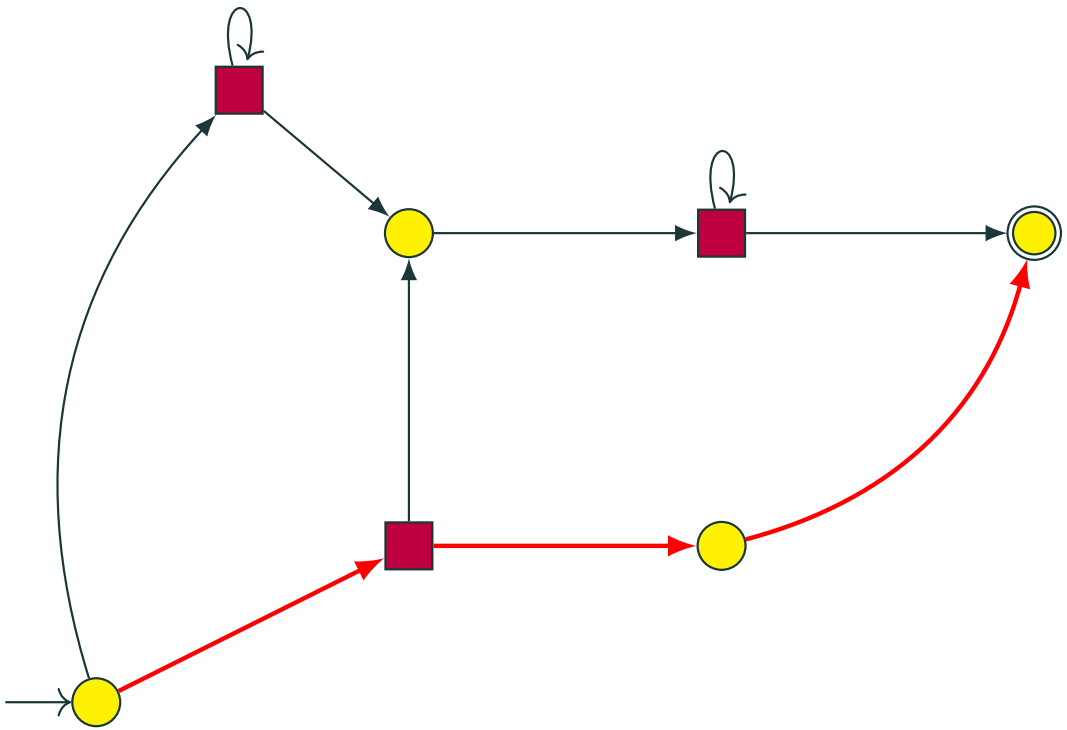

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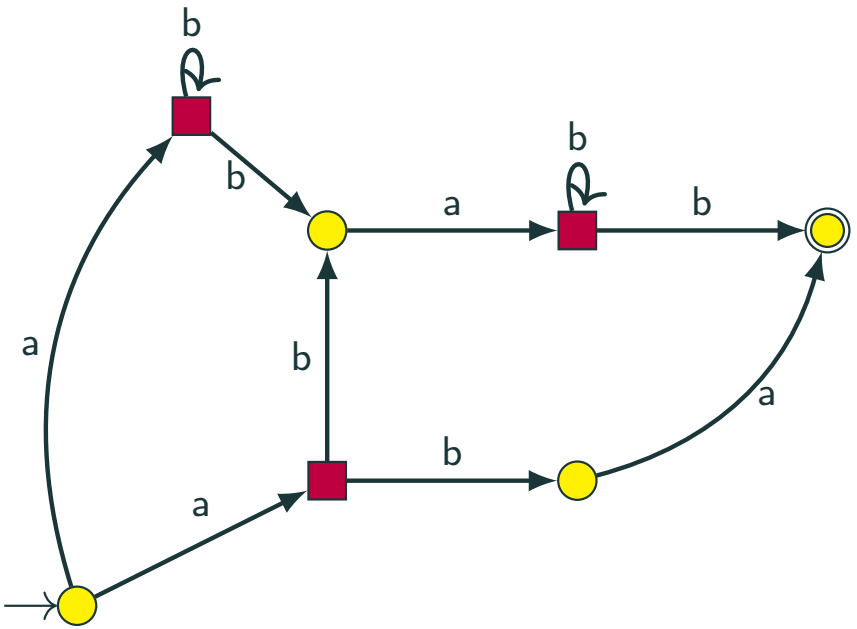
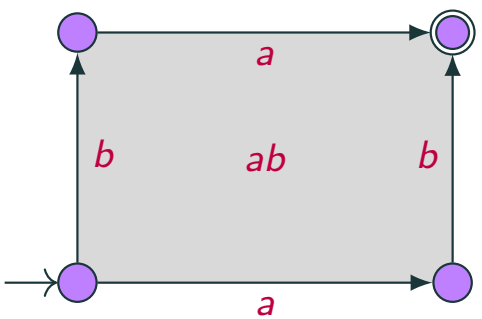
Our goal : Games & Concurrency



+ Parallel events

= Games on HDA ?



Concurrent models : Higher Dimensional Automata

Model

- Distingish non-interleaving and interleaving concurrency : $a||b \neq a.b + b.a$
- Dimension : maximal number of parallel events

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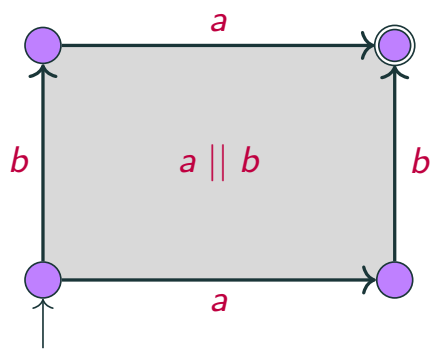

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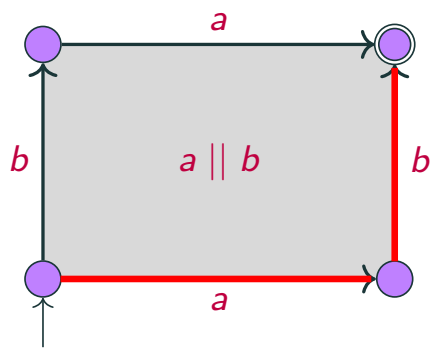
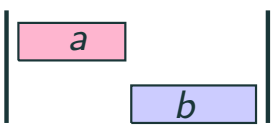


Traces : Interval pomset with interface



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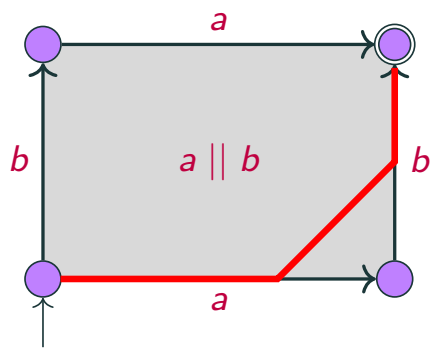

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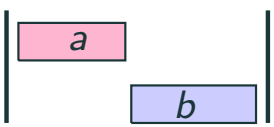

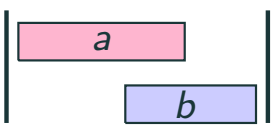

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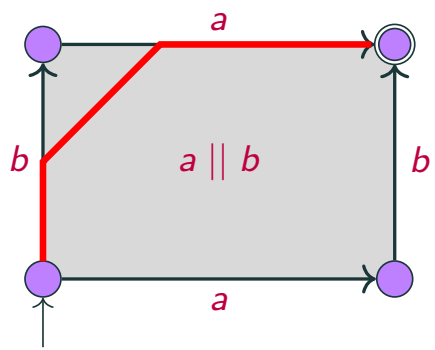

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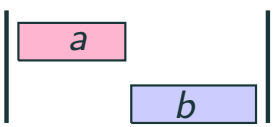

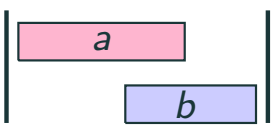

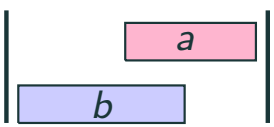

Higher Dimension Automata formally

Conclist : a finite, totally ordered $\Sigma-$labled set *(a list of labeled events)*

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Upper f.m : $\delta^1_A : X[U] \to X[U \backslash A]$ *(terminating ev A)*

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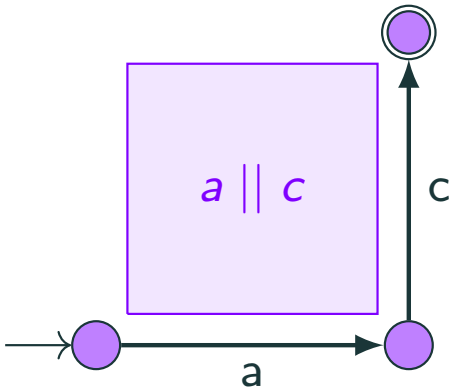
Higher Dimension Automata

A precubical set X, with initial and accepting cells.

Variants and objects around HDA

Partial HDA

- Facemap can be not defined (inclusion of precubical identify is respected)



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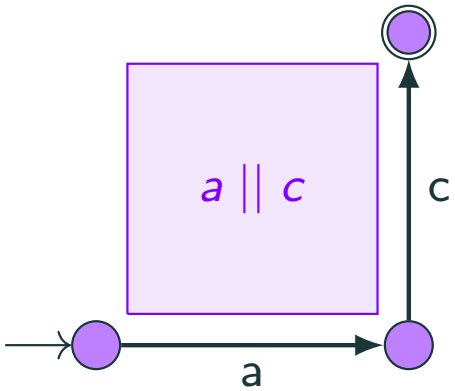
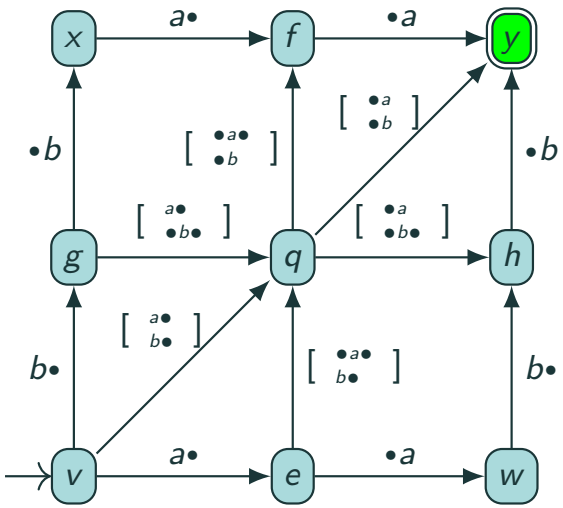


ST-Automata

- Classic automata
- Action are discrete pomsets
- Very expressive
- Too many location



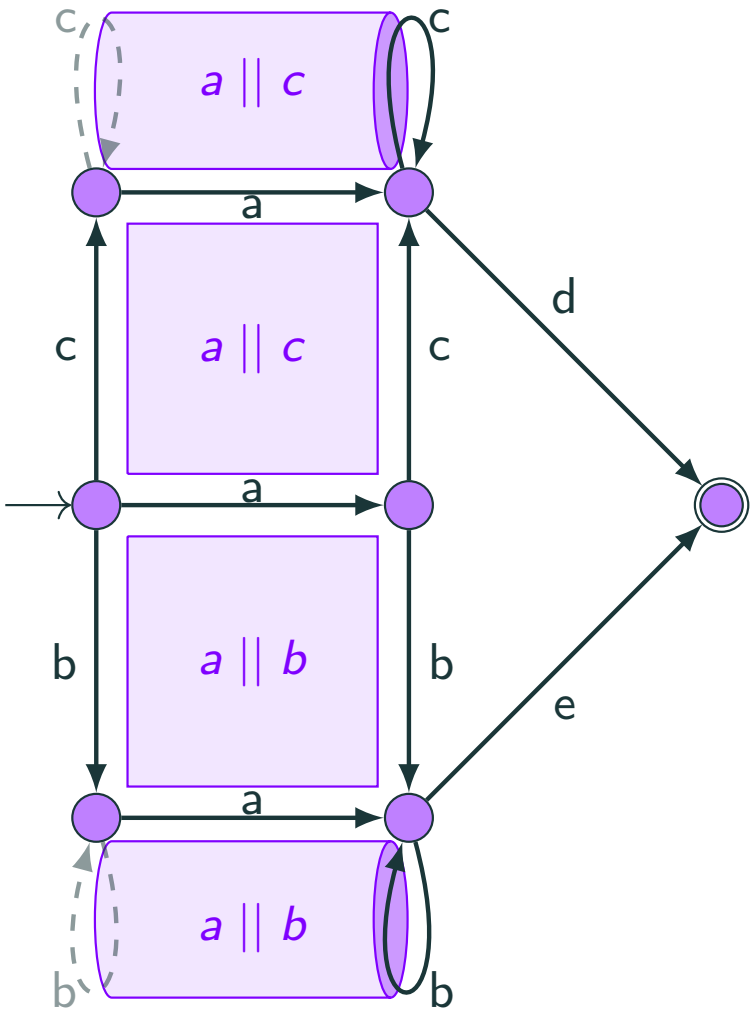
$X[\emptyset] = \{v, w, x, y\}$

$X[a] = \{e, f\}$

$X[b] = \{g, h\}$

$X[ab] = \{q\}$

Example of a game on Higher Dimensional Automata



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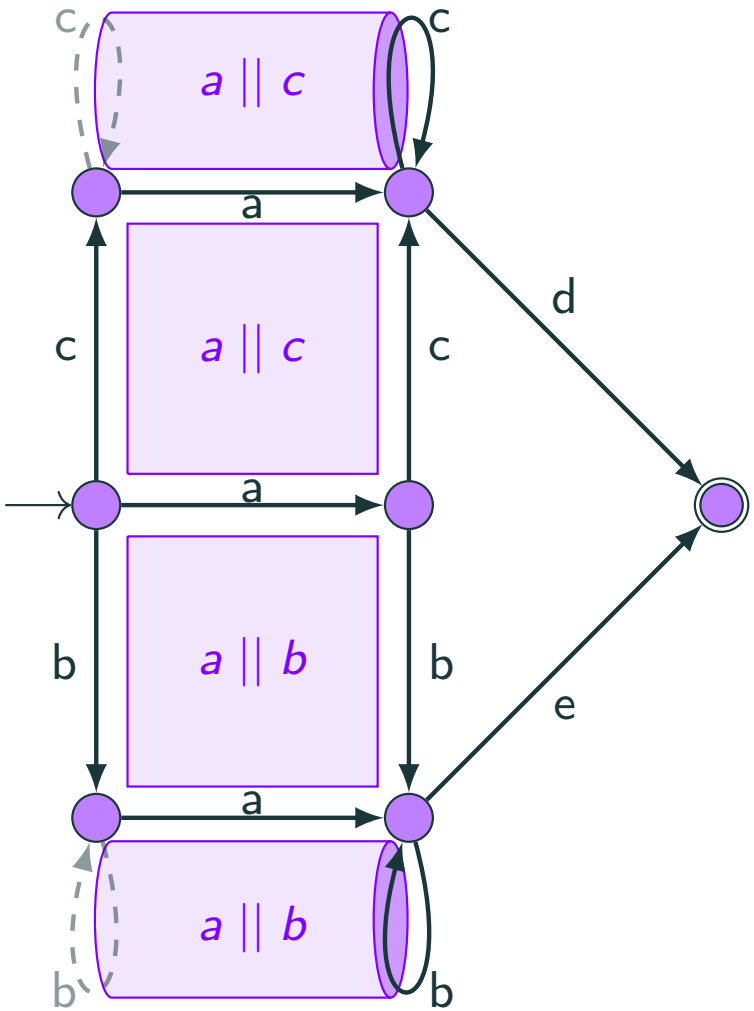

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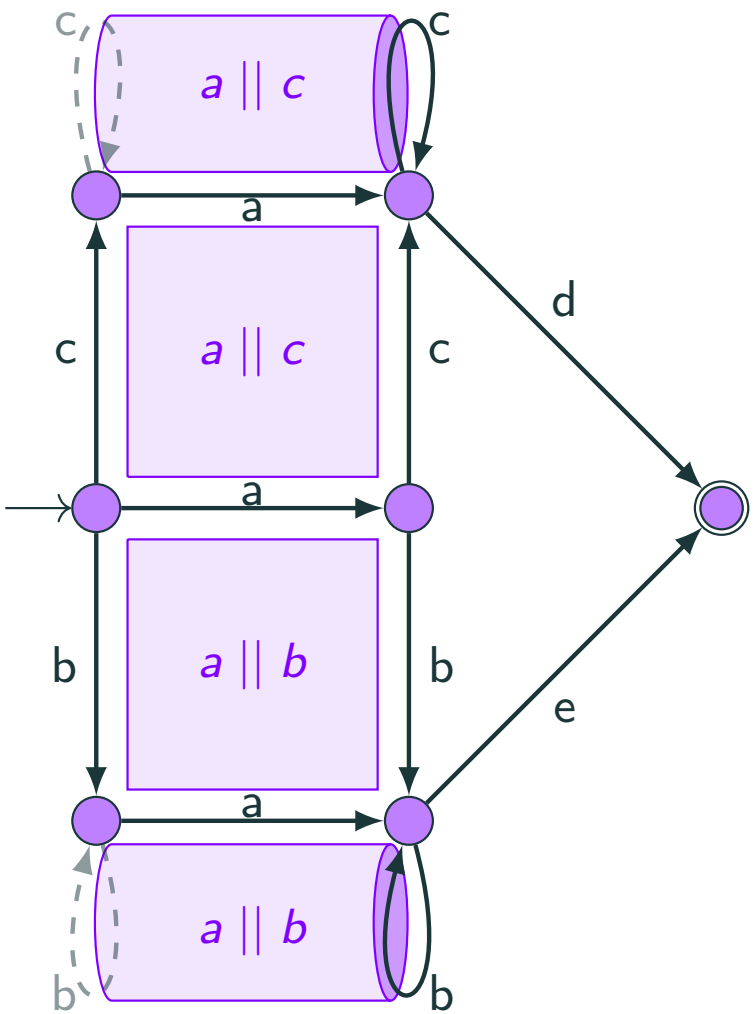

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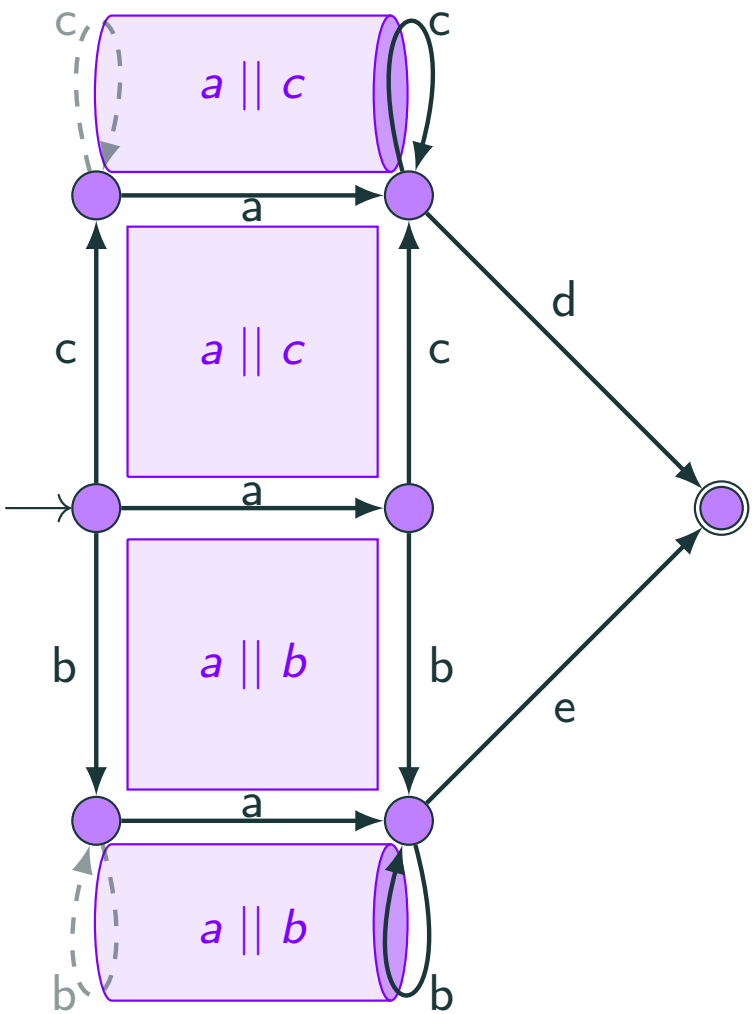

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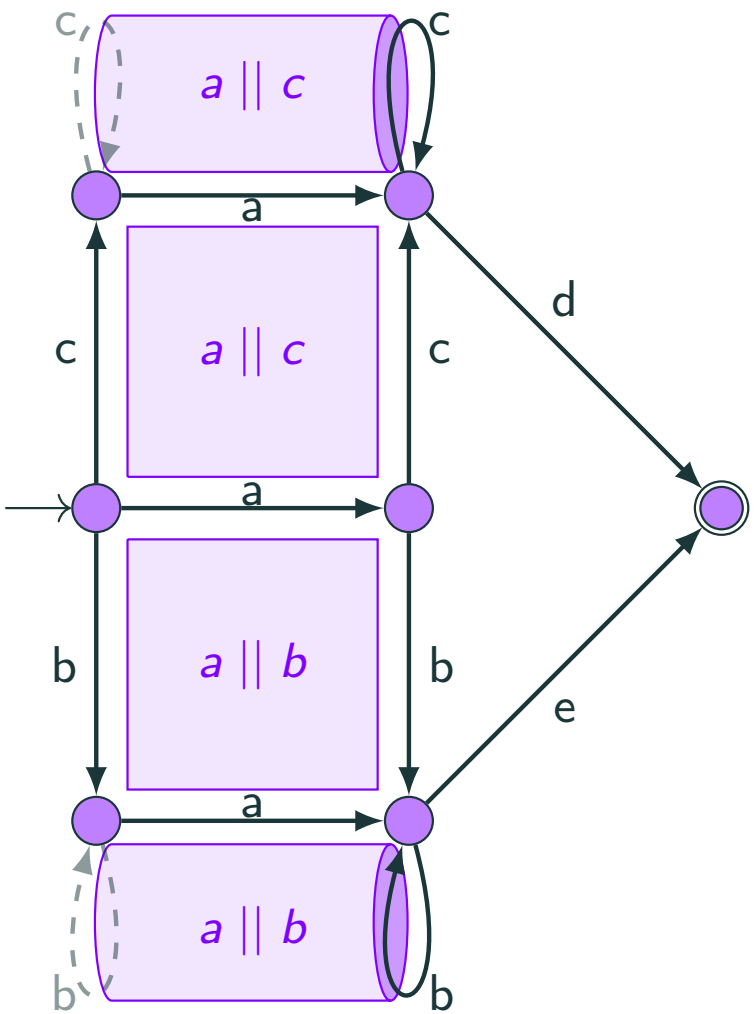

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- Env controls $\Sigma_2 : \{b, c\}$
- Events must be **available**
- **Actions** : S/T only (e.g $a\bullet$ or $\bullet a$)

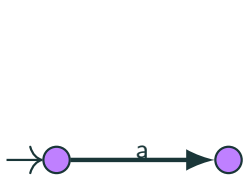
Example on run

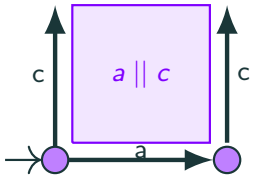
Example on run

$a\bullet$

$c\bullet$

Example on run





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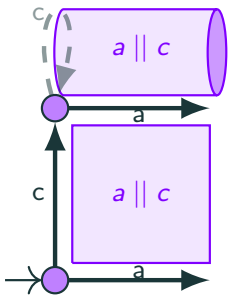

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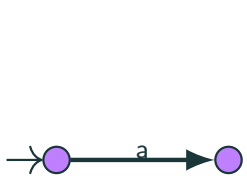

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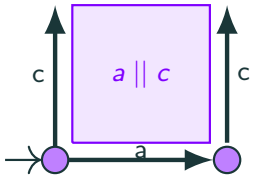

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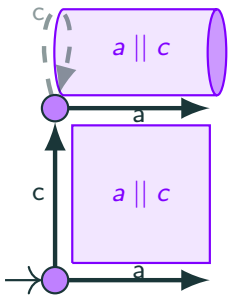
$c\bullet$



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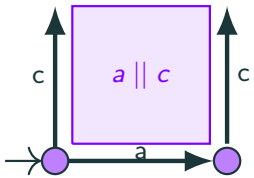

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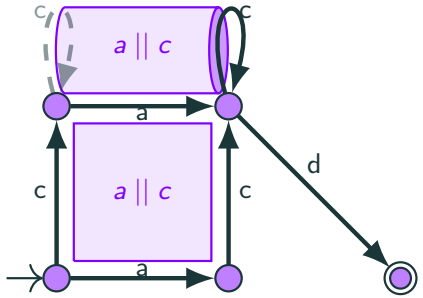

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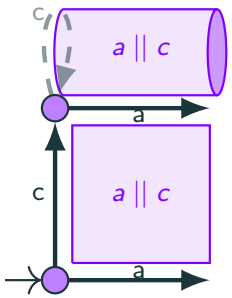

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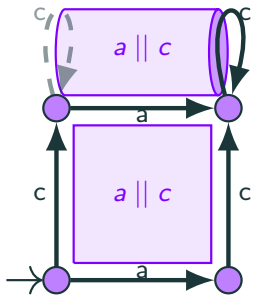

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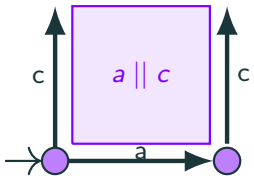

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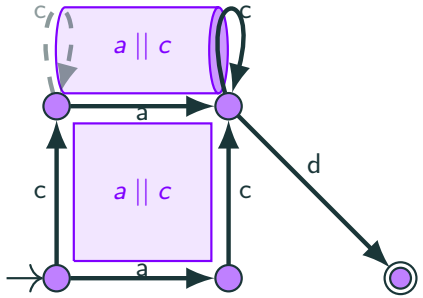

$d\bullet$

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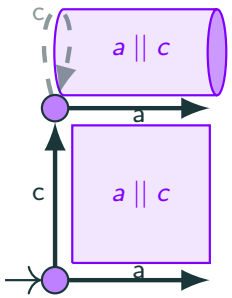

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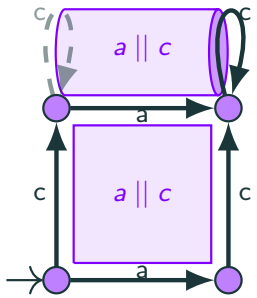

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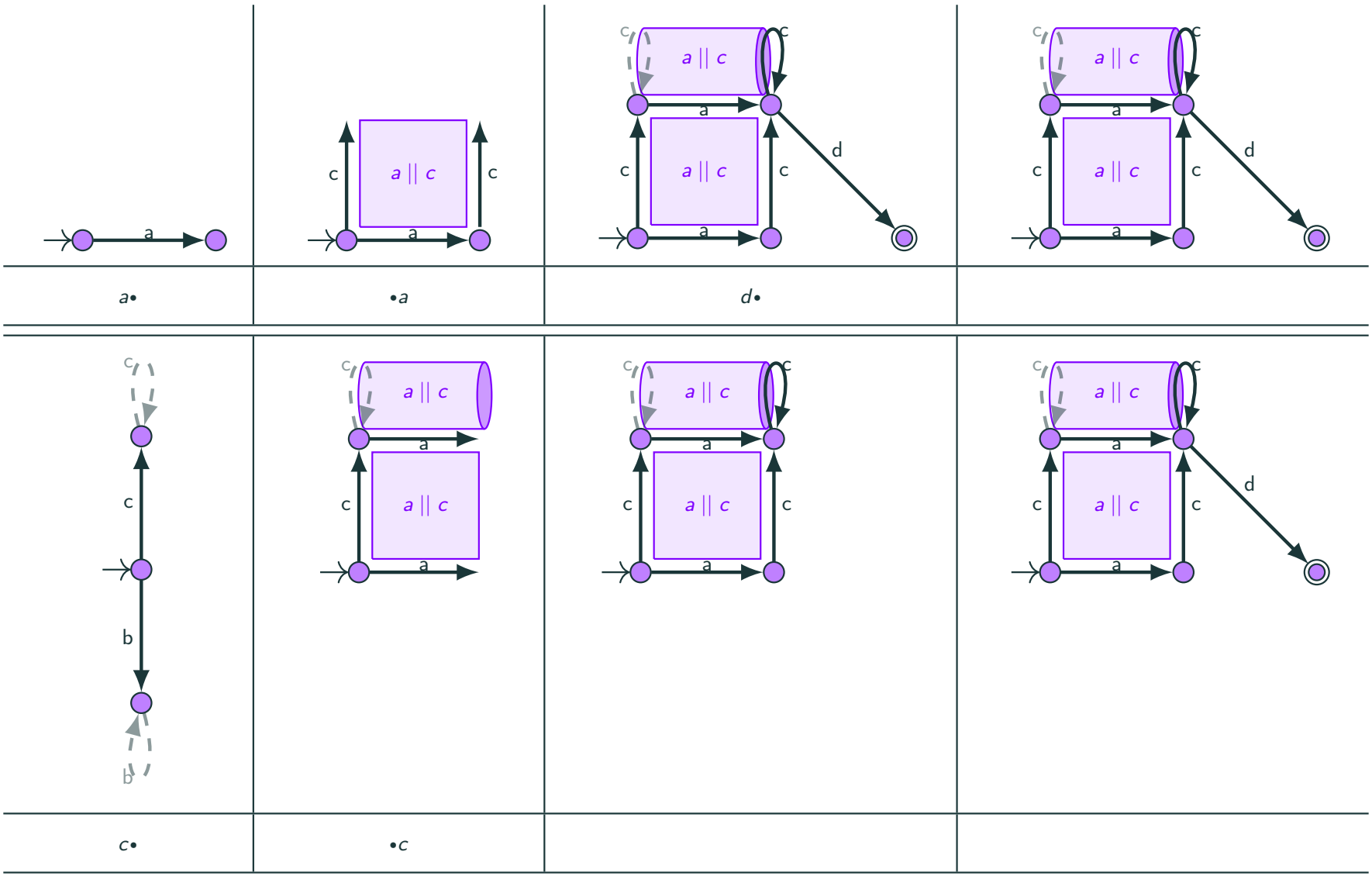

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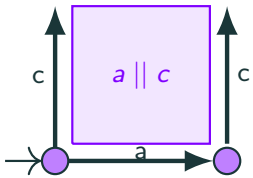

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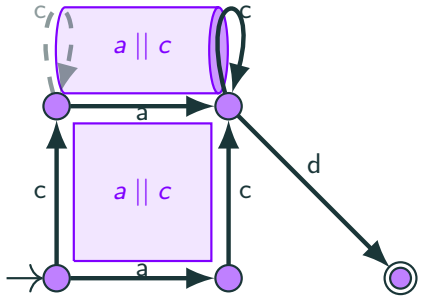

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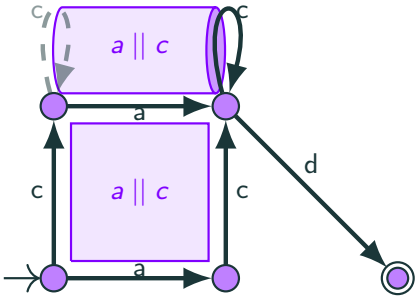

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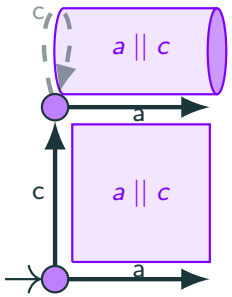

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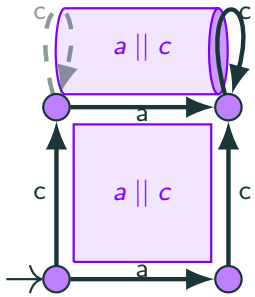

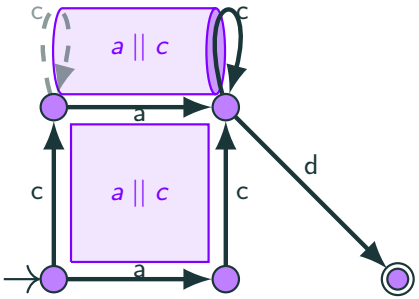

Model a game on a (variant of) graph

A game

- An arena : The model $\mathcal{M}$ + the alphabet of each player Σ_i
- A strategy for each player

Questions

- Action of each players ?
- What is a strategy ?
- Verify optimality of a strategy ?

Issues & Perspectives

Representing view/strategy

- Partial HDA : forbid some cells but not some combination
- "Fragmented" HDA : break precubical identity composition
- ST-Automata : seems to be the right way

Applications

- Classic game problem : Zero-sum, Nash equilibria, multi-strategies ...
- Classic verification problem (synthesis)
- Robustness