# Fan-Causality and Uniform Continuity on Final Coalgebras

Relational and Algebraic Methods in Computer Science (RAMiCS) 2026

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# Compositional Coiteration and Coinduction

# Behavioural Differential Equations

## Streams

- $A^\omega$ is set of streams over an inhabited set $A$
- $\langle \mathrm{hd}, \mathrm{tl} \rangle \colon A^\omega \to A \times A^\omega$ is final coalgebra for $A \times \mathrm{Id} \colon \mathbf{Set} \to \mathbf{Set}$
- $\mathrm{s}_n \in A$ is $n$th element of $\mathrm{s} \in A^\omega$ with $\mathrm{s}_n = \mathrm{hd}(\mathrm{tl}^n(\mathrm{s}))$

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## Specifying Stream via Differential Equations

$$\oplus \colon \mathbb{N}^\omega \times \mathbb{N}^\omega \to N^\omega$$
$$\mathrm{hd}(\mathbf{s} \oplus \mathbf{t}) = \mathrm{hd}(\mathbf{s}) + \mathrm{hd}(\mathbf{t})$$
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- odd well-defined:
  $\mathrm{odd}(\mathbf{s})_{n+1} = (\mathrm{tl}(\mathrm{odd}(\mathbf{s})))_n = \mathrm{odd}(\mathrm{tl}(\mathrm{tl}(\mathbf{s})))_n$
- odd is not causal: $\mathrm{odd}(\mathbf{s})_n = \mathbf{s}_{2n+1}$

# Specifications and Characterisations

## Causal Specifications

- Causal: $n$th approximations of output depends only on $n$th approximation of input
- Recursive specifications and distributive laws limited to causal behaviour[1]
- Causal maps are precisely metric (non-expansive) maps on streams
- Specifications of causal maps can be composed sequentially, in parallel and recursively[2]

[1] Helle Hvid Hansen, Clemens Kupke, and Jan Rutten. "Stream Differential Equations: Specification Formats and Solution Methods". In: *Logical Methods in Computer Science* 13.1 (2017). DOI: 10.23638/LMCS-13(1:3)2017; Jurriaan Rot and Damien Pous. "Companions, Causality and Codensity". In: *Logical Methods in Computer Science* 15.3 (Aug. 2019). DOI: 10.23638/LMCS-15(3:14)2019.

[2] Andreas Abel and Brigitte Pientka. "Wellfounded Recursion with Copatterns: A Unified Approach to Termination and Productivity". In: *ICFP*. 2013, pp. 185–196. DOI: 10.1145/2500365.2500591; Jasmin Christian Blanchette et al. "Friends with Benefits". In: *Programming Languages and Systems*. Ed. by Hongseok Yang. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, 2017, pp. 111–140. ISBN: 978-3-662-54434-1. DOI: 10.1007/978-3-662-54434-1_5.

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## Beyond Causal

- The map $\mathrm{odd}$ is not causal
- Cannot be directly defined by known specification formats/distributive laws

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# Main Goals

- Compositional, recursive specifications of coinductive behaviour beyond causal maps
- Characterise different forms of metric continuity (non-expansive, Lipschitz, uniform) for coinductive computations

# Coalgebra and Coinduction

# Approximation via Descending Chains

- Coinductive objects are final coalgebras for endofunctors $F\colon \mathcal{C} \to \mathcal{C}$
- Final chain: functor $\Phi F\colon \omega^{\mathrm{op}} \to \mathcal{C}$ given by, starting from terminal object $*$,

$$\Phi F = * \xleftarrow{!} F* \xleftarrow{F!} F^2* \xleftarrow{F^2!} \cdots$$

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Example (Streams)

- Infinite sequences $A^\omega$ over a set $A$
- Final coalgebra for functor $S \colon \mathbf{Set} \to \mathbf{Set}$ with $SX = A \times X$
- $\Phi S \cong * = A^0 \longleftarrow A \longleftarrow A^2 \longleftarrow \cdots$

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## Example (Bisimilarity on Streams)

- Final coalgebra for $B \colon \mathrm{Rel}_{A^\omega} \to \mathrm{Rel}_{A^\omega}$ on category of relations ordered by inclusion
- $B(R)(\mathbf{s}, \mathbf{t}) = [\mathrm{hd}(\mathbf{s}) = \mathrm{hd}(\mathbf{t})] \wedge [R(\mathrm{tl}(\mathbf{s}), \mathrm{tl}(\mathbf{t}))]$
- $\Phi B \cong \top = \sim_0 \xleftarrow{\supseteq} \sim_1 \xleftarrow{\supseteq} \sim_2 \xleftarrow{\supseteq} \cdots$ (depth-$n$ equivalence)

# Causality

- Assumption: functor $F\colon \mathcal{C} \to \mathcal{C}$ preserves $\omega^{\mathrm{op}}$-limits
- Final coalgebra as limit: $\nu F = \lim(\Phi F)$
- Limit projections $p_i\colon \nu F \to (\Phi F)_i$
- A morphism $f\colon \nu F \to \nu F$ is *causal* if for all $x, y$ and $i$ commutativity on the left implies commutativity on the right

$$
\begin{array}{ccccc}
 & \nu F & & & \\
{\scriptstyle x}\nearrow & & \searrow{\scriptstyle p_i} & & \\
G & \circlearrowleft & & (\Phi F)_i & \\
{\scriptstyle y}\searrow & & \nearrow{\scriptstyle p_i} & & \\
 & \nu F & & &
\end{array}
\quad\Longrightarrow\quad
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\end{array}
$$

- Write $\mathrm{Caus}(\nu F, \nu F)$ for set of causal maps

# Understanding Causality

**Example (Streams)**

- $f \colon A^\omega \to A^\omega$ causal: outputting $f(\mathbf{s})_n$ requires only elements $\mathbf{s}_0, \ldots, \mathbf{s}_n$
- Point-wise operations causal: $\mathrm{sc}_a \colon \mathbb{R}^\omega \to \mathbb{R}^\omega$ unique solution to

$$\mathrm{hd}(\mathrm{sc}_a(\mathbf{s})) = a \cdot \mathrm{hd}(\mathbf{s}) \qquad \mathrm{tl}(\mathrm{sc}_a(\mathbf{s})) = \mathrm{sc}_a(\mathrm{tl}(\mathbf{s}))$$

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### Example (Completed natural numbers, one-point compactification)

- $\mathbb{N} \cup \{\infty\} \cong \nu N$, where $N \colon \mathbf{Set} \to \mathbf{Set}$ is $N(X) = X + *$

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- Some form of order continuity?

## Causal = Metric = Chain Maps

- Projections induce metric on final coalgebra
- Fix a generator object $G$ and write $\mathrm{El}(\nu F)$ for $\mathcal{C}(G, \nu F)$
- Given $x, y \colon G \in \mathrm{El}(\nu F)$, define

$$d(x, y) = \sup\{2^{-i} \mid i \in \mathbb{N}, p_i \circ x \neq p_i \circ y\}$$

[3] Henning Basold and Tanjona Ralaivaosaona. "Composition and Recursion for Causal Structures". In: *Proc. of CALCO 2023*. Ed. by Paolo Baldan and Valeria de Paiva. Vol. 270. LIPIcs. Schloss Dagstuhl, 2023, 18:1–18:17. ISBN: 978-3-95977-287-7. DOI: 10.4230/LIPIcs.CALCO.2023.18.

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We obtain a characterisation[3]

**Theorem**

1. *A map $f : \nu F \to \nu F$ is causal iff it is non-expansive:* $\mathrm{Caus}(\nu F, \nu F) \cong \mathbf{Met}(\mathrm{El}(\nu F), \mathrm{El}(\nu F))$
2. *Causal maps a exactly chain maps:* $\mathrm{Caus}(\nu F, \nu F) \cong [\omega^{\mathrm{op}}, \mathcal{C}](\Phi F, \Phi F)$

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Morphism in $[\omega^{\mathrm{op}}, \mathcal{C}](\Phi F, \Phi F)$ :

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# Why Chain Morphisms?

- Category $[\omega^{\mathrm{op}}, \mathcal{C}]$ of chains and chain morphisms (functors and natural transformations)
- Inherits products and function space from Cartesian closed $\mathcal{C}$
- Guarded recursion/guarded trace operator for recursive specifications[4]
- For $\mathcal{C} = \mathbf{Set}$ known as "topos of trees"[5]

[4] Sergey Goncharov and Lutz Schröder. "Guarded Traced Categories". In: *Proc. of FOSSACS'18*. Ed. by Christel Baier and Ugo Dal Lago. Vol. 10803. LNCS. Springer, 2018, pp. 313–330. DOI: 10.1007/978-3-319-89366-2_17. URL: https://doi.org/10.1007/978-3-319-89366-2_17.

[5] Lars Birkedal et al. "First Steps in Synthetic Guarded Domain Theory: Step-Indexing in the Topos of Trees". In: *Logical Methods in Computer Science* 8.4 (2012). DOI: 10.2168/LMCS-8(4:1)2012.

# Beyond Causality, Metric Maps and Chain Morphisms

# The Plan

## Continuity

- Metric maps are 1-Lipschitz continuous
- Goal: move to Lipschitz and uniform continuity
- Uniform continuity gives nice guarded recursion

## Causality

- To match this, we generalise causality
- Look-ahead controlled by certain functions called fans

## Chain Morphisms

- Chain morphisms go “straight down”
- Using fans, we can skew the computations

# Beyond Causalilty

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## Fan-Causality

- A fan is an unbounded, monotone map $u\colon \omega \to \omega$: for every $n \in \mathbb{N}$ there is $k \in \mathbb{N}$ with $n \leq u(k)$.

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- Equivalently the map $m_u\colon [0, \infty) \to [0, \infty)$ is a modulus of continuity.

$$m_u(s) = \sup\left\{ 2^{-k} \;\middle|\; k \in \mathbb{N} \text{ with } 2^{-u(k)} \leq s \right\}$$

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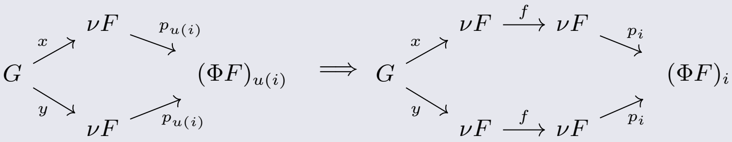
- A morphism $f\colon \nu F \to \nu F$ is $u$-causal for a fan $u$ if for all $x, y$ and $i$ commutativity on the left implies commutativity on the right.



# Fan-Causal = Uniform Causal

**Lemma**

*For any map $f\colon \nu F \to \nu F$, if $f$ is uniformly continuous with modulus of continuity $m$, then there exists a fan $u\colon \omega \to \omega$ such that $f$ is uniformly continuous with modulus of continuity $m_u$.*

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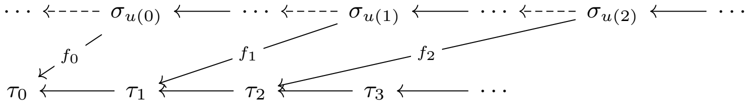
**Theorem**

*A map* $f\colon \nu F \to \nu F$ *is uniformly continuous iff there exists a fan* $u$ *such that* $f$ *is* $u$*-causal.*

# Fanned Chain Morphisms




## Chains and Fan Morphisms

- Denote pre-composition with fan $u$ by $u^*\sigma$: $(u^*\sigma)_n = \sigma_{u(n)}$
- A *$u$-fanned morphism* between chains $\sigma$ and $\tau$ is natural transformation $f\colon u^*\sigma \to \tau$.
- Pair $(u, f)$ is a fanned morphism.
- $\mathbf{Fan}(\mathcal{C})$ category of chains and fanned morphisms

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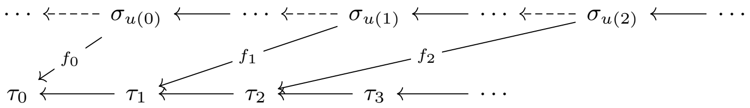


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## Theorem

$\mathbf{Fan}(\mathcal{C})(\Phi F, \Phi F)$ *isomorphic*[6] *to set of fan-causal morphism* $\nu F \to \nu F$

[6] depending on a given section of limit projections

# Guarded Recursion on Fanned Morphisms

## Basic Structure Becomes Lax

- Write $v \leq u$ for point-wise order
- Define $\sigma_{v \leq u} \colon u^* \sigma \to v^* \sigma$ by $(\sigma_{v \leq u})_i = \sigma_{v(i) \leq u(i)} \colon \sigma_{u(i)} \to \sigma_{v(i)}$ ($\sigma$ descending chain)
- $\mathbf{Fan}(\mathcal{C})$ is order enriched: $(u, f) \sqsubseteq (v, g)$ if $v \leq u$ and $f = g \circ \sigma_{v \leq u}$
- Monoidal product $\otimes$ is lax functor
- Monoidal structure has lax natural transformations as unitors etc.
- Later modality $\blacktriangleright \colon \mathbf{Fan}(\mathcal{C}) \to \mathbf{Fan}(\mathcal{C})$ is lax functor

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## Theorem

*There is a guarded trace operator (in a suitable sense)*

$$\mathrm{Tr}_{\sigma,\tau} : \mathbf{Fan}(\mathcal{C})(\blacktriangleright \sigma \otimes \tau, \sigma) \to \mathbf{Fan}(\mathcal{C})(\tau, \sigma)$$

# The End

## Summary

- Generalised causality to fan-causality
- Matches uniform continuity on final coalgebras
- Compositionality via fanned morphisms, but monoidal product is lax for order-enrichment

## Future Work

- Richer metric structure: lift metric on a space to functor and characterise continuity
- In preparation: non-Cartesian categories