

# Morita Equivalence for Kleene Algebras

Luke Serafin

Cornell University

April 6, 2026

1 Kleene Algebras and Modules

2 The Morita Skeleton

3 Tensor Algebras

# Kleene Algebras and Modules

# Kleene Algebras

## Definition (Kozen)

An algebra  $\mathfrak{K} = \langle K, +, \cdot, 0, 1, * \rangle$  of signature  $\langle 2, 2, 0, 0, 1 \rangle$  is a *Kleene algebra* when

- 1 The reduct  $\langle K, +, \cdot, 0, 1 \rangle$  is an additively-idempotent semiring (so  $x + x = x$  for every  $x \in K$ );
- 2 for every  $x \in K$ ,  $x^* = 1 + xx^* = x^*x + 1$ ,
- 3 for every  $a, b, x \in K$ , if  $b + ax \leq x$  then  $a^*b \leq x$ ,
- 4 for every  $a, b, x \in K$ , if  $b + xa \leq x$  then  $ba^* \leq x$ .

For the last two conditions,  $\leq$  is the lattice order for  $\langle K, + \rangle$ , defined by  $x \leq y$  precisely when  $x + y = y$ . The unary operation  $*$  is called the *asterate*.

# Examples of Kleene Algebras

- 1 The algebra of regular languages in a fixed alphabet,
- 2 The algebra of relations on a fixed set, where  $R^*$  is interpreted as the reflexive transitive closure of the relation  $R$ .
- 3 The algebra of sequential processes over a fixed collection of primitive operations.
- 4 The tropical algebra  $\langle \mathbf{R}^{\geq 0} \cup \{\infty\}, \min, +, \infty, 0, * \rangle$ , where  $r^* = 0$  for  $r \in \mathbf{R}^{\geq 0}$ .

# Kleene Modules

- Modules are central to the study of rings.
- The analogous statement is not true for Kleene algebras, but perhaps modules will become more central.

# Kleene Modules

## Definition

For  $K$  a Kleene algebra, an additively-idempotent (left) semiring module  $\langle M, +, 0 \cdot : K \times M \rightarrow M \rangle$  is a *Kleene module* when it satisfies the following for  $a \in K$  and  $m \in M$ : if  $am \leq m$  then  $a^*m \leq m$ .

Right Kleene modules and  $(K, S)$ -bimodules for  $K$  and  $S$  Kleene algebras are defined similarly.

# Examples of Kleene Modules

- 1 The endomorphisms  $\text{End}(K)$  of a Kleene algebra  $K$  form a left  $K$ -module.
- 2 The Lindenbaum-Tarski algebra of a propositional dynamic logic, with  $p[\phi] = [\langle p \rangle \phi]$  for  $\phi$  a sentence in the propositional dynamic logic,  $[\cdot]$  denoting the Lindenbaum-Tarski class of a sentence, and  $p$  an element of the process Kleene algebra.
- 3 For any Kleene  $K$ -module  $M$ ,  $\text{End}(M)$  is a  $K$ -module.

In particular the matrix modules  $\text{End}(K^n)$  for  $n \in \mathbf{N}$  are  $K$ -modules (in fact,  $(K, K)$ -bimodules).

## The Morita Skeleton

# The Morita Category

The Morita skeleton of rings has rings as objects, isomorphism classes of  $(R, S)$ -bimodules as morphisms from  $R$  to  $S$ , bimodules  ${}_R R_R$  as identities, and tensor product as composition.

## Definition

The *Morita skeleton of Kleene algebras* has

- Kleene algebras as objects,
- isomorphism classes  $(K, S)$ -bimodules as morphisms from  $K$  to  $S$ ,
- bimodules  ${}_K K_K$  as identities,
- tensor product of Kleene modules as composition.

In order for this definition to make sense, one must verify that a tensor product exists for Kleene modules and has the expected properties. The usual arguments for rings carry over without special difficulties.

# Morita Equivalence

## Definition

Kleene algebras  $K$  and  $S$  are *Morita-equivalent* if and only if they are isomorphic in the Morita skeleton.

This means that there are Kleene modules  ${}_K M_S$  and  ${}_S N_K$  such that

- ${}_K M_S \otimes {}_S N_K \cong {}_K K_K$ , and
- ${}_S N_K \otimes {}_K M_S \cong {}_S S_S$ .

# Representations and Morita Equivalence

Morita-equivalent Kleene algebras  $K$  and  $S$  have the same representation theory, as we can use the components of an equivalence to change scalars:

- ${}_K P \mapsto {}_S N_K \otimes_K P$ , and
- ${}_S P \mapsto {}_K M_S \otimes_S P$ .

These maps are easily seen to be inverses.

This also enables us to see that Kleene algebras are Morita-equivalent if and only if they have equivalent categories of (left) modules.

# Matrices and Morita Equivalence

## Theorem

*Let  $K$  be a Kleene algebra,  $n$  a positive integer, and  $M$  be the Kleene algebra of  $n \times n$  matrices over  $K$ . Then  $K$  is Morita-equivalent with  $M$ .*

The proof makes use of the dual  $K^{no}$  of the Kleene module  $K^n$ , which is the module of homomorphisms  $K^n \rightarrow K$ .

The key is to show that  $K^n \otimes K^{no} \cong M$  and  $K^{no} \otimes K^n \cong K$ .

## Vectors and Dual Vectors

The proof that  $K^n \otimes K^{n^o} \cong M$  is via showing that the correspondence between  $e_i \otimes e_j^o$  and the matrix with precisely one nonzero entry which is a 1 at index  $ij$  is an isomorphism.

## Vectors and Dual Vectors

The proof that  $K^{n^o} \otimes K^n \cong K$  is illustrated by the following computation.

$$\begin{aligned}(1 \ 0 \ 0) \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= (1 \ 0 \ 0) \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= (1 \ 0 \ 0) \otimes \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= (1 \ 0 \ 0) \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= (1 \ 0 \ 0) \otimes \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\end{aligned}$$

## Vectors and Dual Vectors

$$\begin{aligned} (1 \ 0 \ 0) \otimes \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= (1 \ 0 \ 0) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= (1 \ 1 \ 1) \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

# Full Idempotents and Matrix Algebras

## Theorem

*Kleene algebras  $K$  and  $S$  are Morita-equivalent if and only if for some dimension  $n$  and full idempotent  $e$  in the semiring  $M$  of  $n \times n$  matrices over (without loss of generality)  $K$ ,  $S \cong eMe$  (with the matrix asterate due to Kozen).*

Here an idempotent  $e \in M$  is *full* precisely when  $MeM = M$ . In particular, when two Kleene algebras are Morita-equivalent, one may be viewed as a matrix algebra over the other.

# Applications of Morita Equivalence?

- We have already seen that Morita equivalence implies equivalence of representation theory.
- Given the Lindenbaum-Tarski algebra of a propositional dynamic logic (not necessarily classical!) with modalities from a Kleene algebra  $K$ , the algebra  $K$  may be equivalently replaced by any Morita-equivalent algebra  $S$ . In particular, using matrices in propositional dynamic logic adds no expressive power.
- (Co)homology of Kleene algebrae? Cohomology of semirings has been studied to a degree, and may yield insights about Kleene algebrae.

# Tensor Algebras

# The Tensor Algebra of a Semiring-module

## Definition

Let  $M$  be a semiring-bimodule over a semiring  $S$ . The *tensor algebra* of  $M$  is defined to be

$$T = \bigoplus_{k=0}^{\infty} M^{\otimes k},$$

where  $M^{\otimes k} = M \otimes M \cdots \otimes M$  is the  $k$ -fold tensor product of  $M$ .  
 $T$  naturally has the structure of a semiring.

# The Tensor Algebra of a Kleene-module

Let  $M$  be a  $(K, K)$ -bimodule where  $K$  is a Kleene algebra.

- The tensor algebra  $T$  of  $M$  is an additively idempotent semiring, but we would like a Kleene algebra.
- If  $K$  is  $*$ -continuous, we can fix this by taking the  $*$ -completion of  $T$ , which is adjoint to the forgetful functor from Kleene algebras to additively-idempotent semirings.
- Is there an extension to a Kleene algebra in general?

Thus we have a means of canonically extending an arbitrary Kleene module over a  $*$ -continuous Kleene algebra to a Kleene algebra, via the tensor product.

# Applications of Tensor Kleene Algebras?

- If Lindenbaum-Tarski modules for logics of processes from an algebra  $K$  and tests from a different kind of algebra  $B$  (Boolean, Heyting, MV, etc.) were naturally  $(K, K)$ -bimodules, then the tensor Kleene algebra would provide a way of canonically extending such a module to a Kleene algebra with generalized tests. Unfortunately, Lindenbaum-Tarski modules are only obviously  $(K, K)$ -bimodules when  $K$  is commutative, so more work is needed to make this perspective useful.
- Automata over a Kleene algebra  $K$  may be extended to tensor algebras, and the result includes intriguing examples of multidimensional “automata” which may warrant further exploration.