On Computational Aspects of Cores of Ordered Graphs

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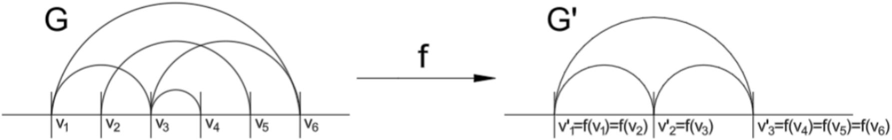
Contents

Introduction

Ordered Graphs and Their Homomorphisms

- An *ordered graph* is a triple $G^{<} = (V, E, \leq_G)$, where (V, E) is a graph, and $\leq_G$ is a total order on V.
- Let $G = (V, E, \leq_G)$ and $G' = (V', E', \leq_{G'})$ be ordered graphs. An *ordered homomorphism from G to G'* is a mapping $f : V \to V'$ that preserves both edges and order. Explicitly, f satisfies
  1. $\{f(u), f(v)\} \in E'$ for all $\{u, v\} \in E$,
  2. $f(u) \leq_{G'} f(v)$ whenever $u \leq_G v$.



Ordered Cores and Chromatic Number

- *Ordered Retraction* of G to H being an ordered homomorphism $f : G \to H$ such that $f(v) = v$ for all $v \in V(H)$.
- If this ordered retraction of G to H exists, we say that G *order-retracts* to H or that H is an *order-retract* of G.
- Notice that, as for digraphs (see Hell and Nešetřil [2004]), if ordered retraction exists, we have ordered homomorphism of G to H and inclusion-ordered homomorphism of H to G, therefore G and H are (ordered) homomorphically equivalent.
- We then define *an Ordered Core* as an ordered graph that does not retract to a proper ordered subgraph.

- The *ordered chromatic number* $\chi^{<}(G)$ of an ordered graph G is equal to

$$\chi^{<}(G) = \min\{i| \text{ there exists an ordered homomorphism } f^{<} : G^{<} \to K_i^{<}\}.$$

Motivation

Motivation - Ordered graphs

- Why *ordered graphs*?
    - Many natural structures come with an inherent linear order.
    - Ordered variants arise in numerous applications, such as physics, data analysis, networked systems, and workflow modeling, whenever the objects are naturally ordered; examples include coordinates, task schedules, gene sequences, etc.
    - Ordered graphs appear naturally in many areas of mathematics and computer science, including extremal combinatorics, Ramsey theory, structural graph theory, and model theory. Pach and Tardos [2006], Balko et al. [2020], Bonnet et al. [2023], Simon [2014]

Motivation - Ordered Homomorphisms

- Why *ordered homomorphisms*?
  - Graph homomorphisms reach across combinatorics, theoretical computer science, and logic, connecting coloring, constraint satisfaction, cores, dualities, and other areas through structure-preserving mappings.
  - Homomorphisms of ordered graphs confirm and complement the above lines of research: they are more restrictive (than standard homomorphisms) but display richness on their own. Kun and Nešetřil [2026], Axenovich et al. [2018]

Motivation - Ordered Cores

- Why *ordered cores*?
  - Studying the core of an ordered graph is essential in studying the ordered homomorphisms, since it gives a canonical representative (of the smallest in its class) for graph homomorphisms (see, e.g., Hell and Nešetřil [1992], Hell and Nešetřil [2004]).
  - Understanding the complexity of these exercises, therefore, becomes crucial in these endeavors, which is the purpose of our study.

Motivation - RAMiCS

- Why *RAMiCS*?
    - This work focuses on a relational structure and structure-preserving maps. We study ordered graphs via ordered homomorphisms, retractions, and cores, placing the paper in the setting of relational methods and abstract structural viewpoints.
    - It is also connected through its computational perspective. Alongside these structural notions, we analyze the complexity of the associated decision problems, linking ordered graph homomorphisms with the algorithmic themes of the conference.

Results

Ordered Core

- We show that ordered cores share the existence and uniqueness properties of cores of unordered graphs.

Theorem

An ordered graph G is a core if and only if there is no ordered homomorphism from G to a proper ordered subgraph of G. Every ordered graph is homomorphically equivalent to a unique ordered core.

Proof.

Outline

- The proof is conceptually analogous to the one in the unordered setting, building on the strategy developed in Hell and Nešetřil [2004].

☐

Complexity of Ordered Graphs Retraction Problem

We consider the following problem, which we call $\mathrm{RET}_<$.

Problem

$\mathrm{RET}_<$

- **Input**: Ordered graph G and a set $X \subseteq V(G)$.
- **Question**: Does there exist a function $f : G \to G[X]$, which is identity on X, and is an ordered homomorphism from G to $G[X]$? (Here $G[X]$ denotes the ordered subgraph of G induced by the set X.)

Theorem

$\mathrm{RET}_<$ *is in* $\mathbf{P}$.

Proof.

Outline

- The statement is established via a reduction of $\mathrm{RET}_<$ to an instance of 2-SAT.

☐

Complexity of Specific Ordered Core Problem

The main result is the computational and parameterized complexity of the following promise problem.

Problem

$\mathrm{CORE}_{<}^{\chi^{<}(G)}$

- **Input**: Ordered graph G with $\chi^{<}(G) = k$.
- **Question**: We want to decide between two cases:
  1. The core of G has k vertices.
  2. G is a core.

Theorem

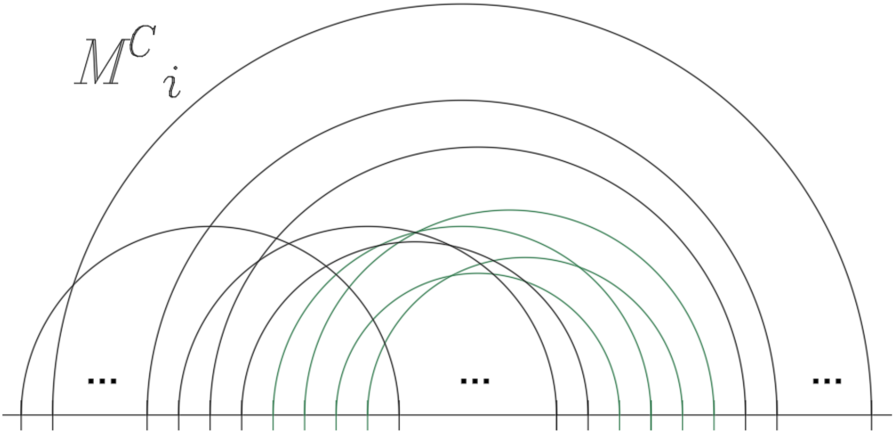
The following hardness results hold. The $\mathrm{CORE}_{<}^{\chi^{<}(G)}$ *problem*

1. *is* $\mathbf{NP}$*-hard,*
2. *is W[1]-hard, when parameterized by* k*.*

Proof Outline

- The proof is a parameter-preserving reduction from $\textsc{Multicolored Clique}$, where from an input graph F with k color classes one constructs an ordered graph G whose structure is designed to simulate the choice of one vertex from each class.
- The key claim is that F has a multicolored clique if and only if G admits an ordered homomorphism to a proper ordered subgraph $H \subset G$, and any such homomorphism is forced to pick exactly one representative vertex from each color class.
- The auxiliary gadgets M_i^C are arranged so that large intervals of vertices can collapse only in one prescribed way, while the copied adjacency pattern from F guarantees that the chosen representatives form a clique exactly when such a homomorphism exists.
- Since the construction is polynomial, satisfies $|V(H)| = \chi^{<}(G)$, and starts from a problem already known to be NP-hard, W[1]-hard, and ETH-hard, the same lower bounds follow for $\mathrm{CORE}_{<}^{\chi^{<}(G)}$.

M_i^C



Complexity of Ordered Core Problems

We then prove the sought after complexity of the following problem.

Problem

$\mathrm{CORE}_<$

- **Input**: Ordered graph G.
- **Question**: Is there a non-surjective homomorphism $G \to G$?

Corollary

$\mathrm{CORE}_<$ *is* $\mathbf{NP}$*-complete and W[1]-hard when parameterized by the number of the vertices of the image ordered proper subgraph of the instance ordered graph.*

Proof.

Outline

- $\mathrm{CORE}_<(\mathcal{G}_{\chi^<(G)})$ is $\mathbf{NP}$-complete, since the same reduction as in Theorem 2 can be applied, while a proposed ordered homomorphism $G \to H, H \subseteq G$ can be verified in polynomial time.

Conclusions

Observations

- The ordered setting preserves the fundamental properties of unordered cores.
- The problem of core for ordered graphs is hard, in a classical as well as parameterized complexity context.
- Core-related problems show different, and sometimes surprising, complexity patterns from $\textsc{Hom}_<$. For instance, for ordered matchings $\textsc{Hom}_<$ appears harder than $\mathrm{CORE}_<$, whereas under the parameter $\chi^<(G)$ the situation reverses: $\textsc{Hom}_<$ is fixed-parameter tractable while $\mathrm{CORE}_<$ is $W[1]$-hard (see Čertík et al. [2025]).
- The complexity of retraction problem differs between ordered graphs and ordered hypergraphs.

Open Problems

Question

Determine for which level i of the W-hierarchy the $\mathrm{CORE}_<$ problem, parameterized by the number of the vertices of the image ordered proper subgraph of the instance ordered graph, is $\mathrm{W}[i]$-complete.

Question

Fully classify the (parameterized) complexity of the same problem, for different classes of the instance ordered graphs.

Question

Extend the (parameterized) complexity results of the same problems, (parameterized by the size of the target structure,) for graphs, oriented graphs, and/or relational systems with partially ordered vertices/elements.

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Thank you

Weighted t-Normalized Satisfiability

Definition (t-Normalized-Satisfiability)

A Boolean formula is called *t-Normalized-Satisfiability* if it can be written in the form of a "product-of-sums-of-products-..." of literals with $t-1$ alterations between products and sums.

For example, $\bigwedge_i \bigvee_{j \in C[i]} v_j$ is 2-Normalized-Satisfiability as in each of the i clauses (product) we can have any (finite) number of variables (or sums of these).

Definition (Weight)

Weight of truth assignment is a number of variables that are set true in t-Normalized-Satisfiability.

Definition (Weighted t-Normalized Satisfiability)

Weighted t-Normalized-Satisfiability is t-Normalized-Satisfiability problem parameterized by weight of truth assignment.

$W[t]$

Definition

$W[t]$
$W[t], t \geq 1$ is the class of all parameterized problems that can be reduced to Weighted t-Normalized-Satisfiability by a parameterized reduction.

Definition

$W[t]$-Hard
$W[t]$-Hard $, t \geq 1$ is the class of all parameterized problems that Weighted t-Normalized Satisfiability can be reduced to by a parameterized reduction.

Definition

$W[t]$-Complete
$W[t]$-Complete $, t \geq 1$ is the class of all parameterized problems that are $W[t]$ and $W[t]$-Hard, $t \geq 1$.

MULTICOLORED CLIQUE

Definition

MULTICOLORED CLIQUE

Let F be the instance graph whose vertex set is partitioned into k subsets $V_1, V_2, \dots, V_k$, each $V_i, i \in [k]$ containing an independent set of vertices. We ask if F has a clique, intersecting every set $V_i, i \in [k]$.

Definition

Promise Problem

The set $L_{\text{YES}} \cup L_{\text{NO}}$ is called the *promise*, where we assume $L_{\text{YES}} \cap L_{\text{NO}} = \emptyset$. An algorithm *solves the promise problem* $(L_{\text{YES}}, L_{\text{NO}})$ if it distinguishes YES-instances from NO-instances, that is:

1. if $x \in L_{\text{YES}}$, it outputs YES;
2. if $x \in L_{\text{NO}}$, it outputs NO;
3. if $x \notin L_{\text{YES}} \cup L_{\text{NO}}$, its behavior is unrestricted.

Parameterized Reduction

Definition (Parameterized Reduction)

Let $L, L' \subseteq \{0,1\}^* \times \mathbb{N}$ be two parameterized decision problems. We say that L reduces to L' by *standard parameterized (many-one-)reduction* if there are functions $k \to k'$ and $k \to k''$ from $\mathbb{N} \to \mathbb{N}$ and a function $(x,k) \to x'$ from $\{0,1\}^* \times \mathbb{N} \to \{0,1\}^*$ such that

1. $(x,k) \to x'$ is computable in $k'' \cdot |(x,k)|^c$ time for some constant c and
2. $(x,k) \in L$ iff $(x',k') \in L'$.