How Can we Compute Coarsest Matrix (Bi)simulations

Roland Glück[1]

[1] Center for Lightweight Production Technology
German Aerospace Center

22st International Conference on Relational and Algebraic
Methods Computer Science
Będlewo, April 4th, 2026

Basics

- matrices - known
- dioid: $(\Delta, \oplus, 0, \otimes, 1)$ with order $\sqsubseteq$ $(x \sqsubseteq y \Leftrightarrow x \oplus y = y)$
- matrices over diods correspond to labeled graphs/transition systems
- (bi)simulations: short reminder to follow
- here: matrix-based characterization of (bi)simulations over dioids

Original Definition of (Bi)simulations

Definition

Given two node labeled graphs $G_1 = (V_1, E_1, L_1)$ and $G_2 = (V_2, E_2, L_2)$, a left-total (total) relation $\mathcal{R} \subseteq V_1 \times V_2$ is called a *simulation* (*bisimulation*) for G_1 and G_2 if it fulfills the following conditions:

- $v_1 \mathcal{R} v_2 \Rightarrow L_1(v_1) = L_2(v_2)$
- $v_1 \mathcal{R} v_2 \wedge (v_1, w_1) \in E_1 \Rightarrow \exists w_2 : w_1 \mathcal{R} w_2 \wedge (v_2, w_2) \in E_2$
- $(v_2 \mathcal{R}^\circ v_1 \wedge (v_2, w_2) \in E_2 \Rightarrow \exists w_1 : w_2 \mathcal{R}^\circ w_1 \wedge (v_1, w_1) \in E_1)$

Relational Characterization of (Bi)simulations

- $v_1 \mathcal{R} v_2 \Rightarrow L_1(v_1) = L_2(v_2)$: remains unchanged
- $v_1 \mathcal{R} v_2 \wedge (v_1, w_1) \in E_1 \Rightarrow \exists w_2 : w_1 \mathcal{R} w_2 \wedge (v_2, w_2) \in E_2$ turns into $\mathcal{R}^\circ; E_1 \subseteq E_2; \mathcal{R}^\circ$
- $v_2 \mathcal{R}^\circ v_1 \wedge (v_2, w_2) \in E_2 \Rightarrow \exists w_1 : w_2 \mathcal{R}^\circ w_1 \wedge (v_1, w_1) \in E_1$ into $\mathcal{R}; E_2 \subseteq E_1; \mathcal{R}$
- for auto(bi)simulations on $G = (V, E, L)$: $\quad \mathcal{R}^\circ; E \subseteq E; \mathcal{R}^\circ$ $(\mathcal{R}; E \subseteq E; R^\circ)$

0-1-matrices

- 0-1-matrices over a dioid are isomorphic to relations
- algebraic counterparts of relational terms
- $A \in \{0, 1, \}^{n \times n}$ is transitive if $A \otimes A \sqsubseteq A$
- A is symmetric if $A^t = A$
- A is reflexive if $id \sqsubseteq A$
- …

Matrix-Based (Bi)simulations

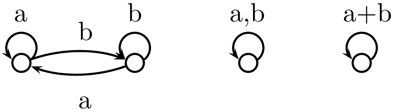
Definition

Given a matrix $A \in \Delta^{n \times n}$, a left-total (total) 0-1-matrix $R \in \{0,1\}^{n \times n}$ is called a *matrix simulation* (*matrix bisimulation*) if it fulfills the following condition(s):

- $R^t A \sqsubseteq A R^t$
- $(RA \sqsubseteq AR)$

matrix (bi)simulations are closed under sum, product (and transposition), hence there is a greatest matrix (bi)simulation

Classical (Labeled) Bisimulation vs. Matrix-Based Ones



$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} a+b \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix} = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} a & b \\ a & b \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \sqsubseteq \begin{pmatrix} a+b & a+b \end{pmatrix} = \begin{pmatrix} a+b \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Computing Greatest (Bi)simulations

- greatest matrix (bi)simulation is computable:
- simply check all 0-1-matrices of dimension $n \times n$
- of course no reasonable method
- how to do this?
- Tarjan/Paige compute coarsest bisimulations in $\mathcal{O}(|E| \cdot log(|V|))$ time (for connected graphs)
- coarsest simulation computable in $\mathcal{O}(|E| \cdot |V|)$ time (for connected graphs)

Solution

Your Algorithm Goes here