Some Aspects of Linear Algebra over Boolean Lattices

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DLR

Basics

- lattice $\mathcal{B} = (M, \sqsubseteq, \sqcup, \sqcap)$ known
- Boolean lattice $\mathcal{B}$ = lattice + complement $\bar{\cdot}$
- for conveniece, everything will be complete
- top element $\top$, bottom element $\bot$
- vectors written as v, w
- constructions from linear algebra with $\sqcup$ acting as addition and $\sqcap$ as multiplication

Span, Generating System, Basis

Definition

Let $S = \{\mathrm{v}^1, \mathrm{v}^2, \ldots, \mathrm{v}^m\}$ be a finite set of m vectors from $\mathcal{B}^n$ and let λ_1, λ_2, …, λ_m be elements of $\mathcal{B}$. A vector of the form $\bigsqcup_{i=1}^{m} \lambda_i \mathrm{v}^i$ is called a *linear combination* of S. The set of all linear combinations of S (the *span* of S) is denoted by $sp(S)$.

Definition

Given a span $S \subseteq \mathcal{B}^n$ we call a finite set T a *generating system of S* if $sp(T) = S$ holds. A generating system is called a *basis of S* if it has minimal cardinality among all generating systems of S.

- without S given, these terms refer to $S = \mathcal{B}^n$

Characterization of Bases

Lemma

A set $S = \{\mathrm{v}^1, \mathrm{v}^2, \ldots, \mathrm{v}^m\}$ *of vectors from* $\mathcal{B}^n$ *is a generating system iff for every* i *with* $1 \leq i \leq n$ *the vector* e^i *is contained in* $sp(S)$*.*

Proof sketch:

- $\Rightarrow$: clear by definition of generating system
- $\Leftarrow$: construct all e^i as linear combinations from S, then write a given v as linear combination of all e^i, then simplify

Inverse Matrices

Definition

Given a matrix A $\in \mathcal{B}^{m\times n}$ we say that a matrix B $\in \mathcal{B}^{n\times m}$ is a *right inverse of* A if AB = I holds. Analogously we define for a matrix A a *left inverse* by the equation BA = I. For a quadratic matrix A we call a matrix B an *inverse* of A if AB = I = BA holds.

Characterization of Right Invertibility

Lemma

Let $S = \{\mathrm{v}^1, \mathrm{v}^2, \dots, \mathrm{v}^m\}$ be a set of vectors and let $\mathrm{A} =_{df} (\mathrm{v}^1|\mathrm{v}^2|\dots|\mathrm{v}^m)$ be the matrix constituted by the elements from S as columns. Then S is a generating system iff A *has a right inverse.*

Proof sketch:

$\Rightarrow$: write $\mathrm{e}^i = \bigsqcup_{j=1}^{m} \lambda_{ij}\mathrm{v}^j$, define L by $\mathrm{L}_{ij} =_{df} \lambda_{ij}$, then $\mathrm{AL} = \mathrm{I}$ holds

$\Leftarrow$: $\mathrm{AB} = \mathrm{I}$ implies $\mathrm{e}^i = \bigsqcup_{j=1}^{m} \mathrm{B}_{ij}\mathrm{v}^j$, apply previous lemma

Exploiting Atomicity

- $a \neq \bot$ is an *atom* if $b \sqsubset a \Rightarrow b = \bot$
- a atom $\Rightarrow a \sqsubseteq x \sqcup y \Leftrightarrow a \sqsubseteq x \vee a \sqsubseteq y$
- a atom $\Rightarrow a \sqsubseteq \top$ (presupposes nontriviality of $\mathcal{B}$)
- atomic lattice: every element x can be written as supremum of the set of atoms below x
- decomposition into supremum of atoms is unique

Theorem

Let $\mathcal{B}$ be an atomic lattice. Then every basis of $\mathcal{B}^n$ has cardinality n.

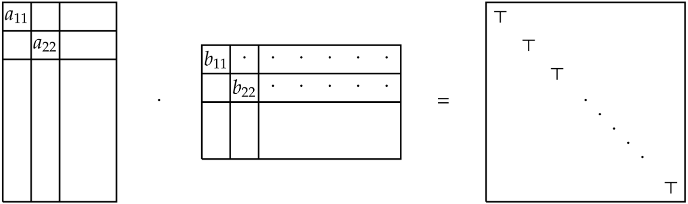
Proof

Proof sketch:

- $\{\mathrm{e}^i \mid 1 \leq i \leq n\}$ is generating system
- hence it suffices to show that there is no generating system with lower cardinality
- assume existence of generating systen with cardinality $m < n$
- construct an $n \times m$-matrix A out of these vectors
- by previous lemma, A has a right inverse B
- consider arbitrary atom a
- $\mathrm{I}_{11} = \top \Rightarrow \exists i : a \sqsubseteq \mathrm{A}_{1i} \wedge a \sqsubseteq \mathrm{B}_{i1}$
- swapping columns 1 and i of A and rows 1 and i of B does not affect $\mathrm{AB} = \mathrm{I}$
- hence w.l.o.g. assume $a \sqsubseteq \mathrm{A}_{11} \sqcap \mathrm{B}_{11}$

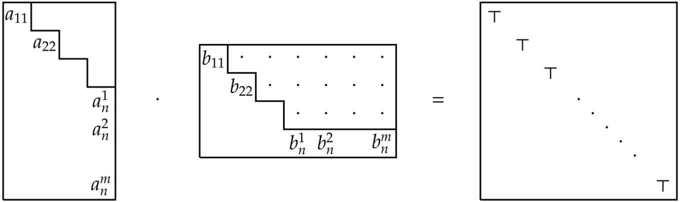
Proof (continued)

- $(AB)_{ij} = \perp$ for $i > 1$
- hence B_{1i} must not contain a for $i > 1$
- analogous argument for $A_2 2$ and B_{22}



Proof (continued)

- repeat this argument inductively
- leads to following situation (dotted areas do not contain a)
- a_n^i is located at position $(n-m+i, n)$ and b_n^i at position $(m, n-m+i)$



- all a_n^i and b_n^i have to contain a
- but then also $(\mathrm{AB})_{(n-1)n}$ has to contain a
- but has to be $\perp$

Theorems, Theorems, Theorems

Theorem

Let A *be a right invertible quadratic matrix over an atomic lattice. Then* A *is invertible and has the unique inverse* A^T*.*

Theorem

A matrix A *over an atomic lattice is diagonalizable iff it is a diagonal matrix.*

Theorem

Let A *be an invertible matrix and* y *a vector over an atomic lattice. Then the equation* $\mathrm{Ax} = \mathrm{y}$ *has the unique solution* $\mathrm{x} = \mathrm{A}^T\mathrm{y}$*.*

Open Questions

- computation of bases
- generalizition from atomic lattices to arbitrary lattices
- possible applications in computer science