Complexity of Equational Theories for Relational and Language Action Lattices

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Max Kanovich · Stepan L. Kuznetsov · Andre Scedrov

Introduction

- The study of substructural logical systems is a vivid area in the research domain of non-classical logics, finding various applications in computer science, linguistics, *etc.*
- We consider **infinitary action logic** $\mathbf{ACT}_\omega$, which is the extension of the (multiplicative-additive) Lambek calculus with Kleene star.
- This logic has two natural families of models—models on formal languages, in the lines of linguistic motivations, and models on algebras of binary relations, formalising the 'action' intuition.
- We prove that the logics of each of these classes of models are Π_1^1-complete (while $\mathbf{ACT}_\omega$ itself is only Π_1^0-complete).
- This solves a question left open by Buszkowski (RelMiCS 2006).

Π^1_1 vs. Π^0_1

- It is worth noticing that the complexity gap between Π^0_1 and Π^1_1 is enormous.
- Π^0_1 is located at the very bottom of the first-order arithmetical hierarchy (one unbounded universal quantifier over natural numbers).
- It is dual to Σ^0_1, the class of enumerable sets, and the typical Π^0_1-complete problem is non-halting for Turing machines.
- In contrast, Π^1_1 belongs to the second-order, or analytical, hierarchy, allowing one universal quantifier over **sets** of natural numbers.
- Π^1_1 properly includes the whole arithmetical and hyper-arithmetical hierarchy.

Residuated Kleene Lattices

Definition

A **residuated Kleene lattice** (RKL) is a structure $(A, \preccurlyeq, \wedge, \vee, \cdot, 0, 1, \backslash, /, ^*)$, where

1. $(A, \preccurlyeq, \wedge, \vee)$ is a lattice;
2. $(A, \cdot, 1)$ is a monoid, 0 being its zero;
3. $a \preccurlyeq c \,/\, b \iff a \cdot b \preccurlyeq c \iff b \preccurlyeq a \setminus c$;
4. $a^* = \min_{\preccurlyeq} \{b \mid 1 \vee a \cdot b \preccurlyeq b\}$.

[Pratt 1991; Kozen 1994]

Definition

An RKL is called **$*$-continuous**, if $a^* = \sup_{\preccurlyeq} \{a^n \mid n \in \mathbb{N}\}$ for all $a \in A$.

Historical Remarks

- Divisions (residuals) related to the partial order: Krull [1924]; Ward, Dilworth [1939]; Lambek [1958].
- Kleene iteration (star): Kleene [1956].
- The notion of RKL was introduced in [Kozen 1994] (earlier, [Pratt 1991] introduced its fragment without $\wedge$).
- Kleene star is the basis for **regular expressions**, which are used as patterns of formal languages.
- Residuated structures are the algebraic basis for **Lambek categorial grammars**, which are used for modelling natural language syntax.
- One of the standard examples of RKLs are **algebras of formal languages**; they are $*$-continuous.

Historical Remarks

- Pratt [1991] defines **action algebras** (which are RKLs, but without the meet operation $\wedge$) by the following conditions on Kleene star:

$$1 \vee (a^* \cdot a^*) \vee a \preccurlyeq a^*; \qquad a^* \preccurlyeq (a \vee b)^*; \qquad (a \setminus a)^* \preccurlyeq a \setminus a.$$

  - The principle $(a \setminus a)^* \preccurlyeq a \setminus a$ is called **"pure induction,"** being the simplest form of induction axiom.
- Pratt's conditions are equivalent to $a^* = \min_{\preccurlyeq} \{b \mid 1 \vee a \cdot b \preccurlyeq b\}$, which we use in our definition of RKL.
- The advantage of "pure induction" is that it is an equational condition, while usual fixpoint definitions result in quasi-equation (Horn formulae).

Historical Remarks

- In the absence of divisions, Kleene algebras are defined using quasi-equations [A. Salomaa 1966; Kozen 1990]:

$$x \cdot a \preccurlyeq x \Rightarrow x \cdot a^* \preccurlyeq x; \qquad a \cdot x \preccurlyeq x \Rightarrow a^* \cdot x \preccurlyeq x.$$

- In the setting of Kleene algebra, without divisions, a finite equational axiomatisation is impossible [Redko 1967; Conway 1971]. In particular, there is no such axiomatisation for equivalence of regular expressions.
- This is a significant difference between Kleene algebras and RKLs.
- Another difference is the existence of anomalous Kleene algebras, where $\sup_{\preccurlyeq} \{a^n \mid n \in \mathbb{N}\}$ exists, but does not equal a^* [Conway 1971]. In RKLs, this is impossible: once all such suprema exist, the RKL is $*$-continuous.

Definition

An **algebra of formal languages** is an RKL $(\mathscr{P}(\Sigma^*), \subseteq, \cap, \cup, \cdot, \varnothing, \{\varepsilon\}, \backslash, /, {}^*)$, where:

1. $M \cdot N = \{uv \mid u \in M, v \in N\}$;
2. $M \setminus N = \{v \mid (\forall u \in M)\, uv \in N\}$, $N \,/\, M = \{v \mid (\forall u \in M)\, vu \in N\}$;
3. $M^* = \bigcup_{n=0}^{\infty} M^n$ (where $M^0 = \{\varepsilon\}$).

- Algebras of formal languages are always $*$-continuous, and their lattice structure is distributive.
- ε denotes the empty word.

Algebras of Binary Relations

- Another example of RKLs: algebras of binary relations.

Definition

An **algebra of binary relations** is an RKL $(\mathscr{P}(W \times W), \subseteq, \cap, \cup, \circ, \emptyset, \delta, \backslash, /, ^*)$, where:

1. $R \circ S = \{(x, z) \mid (\exists y \in W)((x, y) \in R, (y, z) \in S)\}$ (relational composition);
2. $R \backslash S = \{(y, z) \mid R \circ \{(y, z)\} \subseteq S\}$; $S / R = \{(x, y) \mid \{(x, y)\} \circ R \subseteq S\}$;
3. R^* is the reflexive-transitive closure of R;
4. $\delta = \{(x, x) \mid x \in W\}$ (the diagonal relation, also called identity).

- Algebras of binary relations are also $*$-continuous and distributive.

Equational Theories

- The **equational theory** of a given class of structures is the set of equalities of terms (in the given language), which are generally valid on this class.
  - Our main (and only) predicate symbol will be inequality $\preccurlyeq$, so a more accurate term would be "**inequational** theory." However, in the presence of $\vee$ it is essentially the same: $a \preccurlyeq b \iff a = a \vee b$.
- In the restricted language of $\cdot, \vee, {}^*$ (Kleene algebras), this is the classical **regular expression equivalence** problem.
  - For example, $(a \vee b)^* = a^* \cdot (b \cdot a^*)^*$.
  - In this case, the equational theory is the same for the following classes of structures: all Kleene algebras; $*$-continuous Kleene algebras; algebras of formal languages.
  - This theory is algorithmically decidable, being PSPACE-complete [Stockmeyer, Meyer 1973].

Equational Theories of RKLs

- In the richer language of RKLs (in the presence of divisions) these theories differ.
  - Example: $(a \wedge b \wedge (a \setminus b) \wedge (a \,/\, b))^+ \preccurlyeq a$ (where $A^+ = A \cdot A^*$) is generally valid on $*$-continuous RKLs, but not on all RKLs.
- Their algorithmic complexity is also different.

Theorem

The equational theory of $*$-continuous RKLs is Π^0_1-complete. [Buszkowski, Palka 2007]

Theorem

The equational theories of all RKLs is Σ^0_1-complete. [Kuznetsov 2021]

- From the syntactic point of view, these theories are axiomatised by infinitary action logic $\mathbf{ACT}_\omega$ and action logic $\mathbf{ACT}$, respectively.

Equational Theories for Languages and Relations

- We study the complexity of equational theories for the aforementioned concrete classes of RKLs.
- Equational theories of algebras of formal languages and of algebras of binary relations strictly extend the equational theory of all $*$-continuous RKLs (in other words, $\mathbf{ACT}_\omega$ is incomplete w.r.t. these concrete classes of models).
- A counter example can be constructed using the 'unit' constant:

$$1 \wedge a \preccurlyeq (1 \wedge a) \cdot (1 \wedge a).$$

  - This is a particular case of the structural rule of **contraction**. In the presence of contraction, complexity can raise significantly.
  - There is also an example which separates these two theories: $(1 \wedge a) \cdot b \preccurlyeq b \cdot (1 \wedge a)$ is generally valid for languages, but not for relations.

Complexity of Equational Theories

- Buszkowski [2006] proved the lower bound, Π^0_1-hardness, for equational theories of algebras of languages and algebras of relations.
- He also conjectured the corresponding upper bound, but proved it only for a special class of algebras of languages, where all languages are **regular.**
  - In this case the quantifier “for all models” becomes arithmetical (“for all tuples of regular expressions”).
- In the general case, only a much higher upper bound is easily achieved: Π^1_1.
- **We prove that this upper bound is exact!**

Complexity of Equational Theories

Theorem

The equational theory of algebras of formal languages (in the language of RKLs) is Π^1_1*-complete.*

Theorem

The equational theory of algebras of binary relations (in the language of RKLs) is Π^1_1*-complete.*

Representation of Horn Theories

- The main idea of our proofs is to embed the more expressive **Horn theory** into the equational one, using specifically designed constructions involving constants 0 and 1.
  - The Horn theory consists of generally valid formulae of the form $\left(A_1 \preccurlyeq B_1 \,\&\, \ldots \,\&\, A_n \preccurlyeq B_n\right) \Rightarrow C \preccurlyeq D$.
  - Example: $(a \cdot b \preccurlyeq b \cdot a \,\&\, b \cdot a \preccurlyeq a \cdot b) \Rightarrow (a \vee b)^* \preccurlyeq a^* \cdot b^*$.
- It is known that the Horn theory of the class of $*$-continuous RKLs has the necessary complexity.

Theorem

The Horn theory of the class of all $*$-continuous RKLs is Π^1_1-complete. [Kozen 2002; Kuznetsov, Speranski 2022]

Complexity of Horn Theories

- However, this is insufficient for our needs: in the full language of RKLs the Horn theories (and even equational ones) for algebras of languages and algebras of relations do not coincide with the theory of the class of all $*$-continuous RKLs.
- Let us define **iterative divisions** [Kuznetsov, Ryzhkova 2020; Sedlár 2020]: $A \mathbin{\backslash\backslash} B = A^* \setminus B$ and $B \mathbin{//} A = B \mathbin{/} A^*$.

Theorem

In the language of $\backslash$, $/$, $\backslash\backslash$, $//$, $\wedge$, the Horn theories of the following classes coincide:

1. *all $*$-continuous RKLs;*
2. *algebras of formal languages;*
3. *algebras of binary relations.*

[Kuznetsov, Ryzhkova 2020; Kuznetsov 2024]

Complexity of Horn Theories

Theorem

The fragment of the Horn theory of the class of all $*$-continuous RKLs, in the language of $\backslash, /, \backslash\backslash, //, \wedge$, is Π^1_1-complete.

- By the previous theorem, the class of all $*$-continuous RKLs can be replaced by the class of algebras of languages or the class of algebras of relations.
- The proof combines ideas from [Kozen 2002] (encoding of the noetherianity problem for a computable binary relation) and [Buszkowski 1982] (which replaces product by division and Kleene iteration by iterative division, in the encoding of Turing machines).

Representation of Horn Theories

- The final step is the embedding of the Horn theory into the equational one, for a given concrete class of models.
- Let us call an operation $A \mapsto \widehat{A}$ a **"classicalisation"** of formulae on the given class of models, if for any model $\mathcal{M} = (A, \alpha)$ of this class the following holds (α being the interpretation function):

$$\mathcal{M} \vDash 1 \preccurlyeq A \quad \Rightarrow \quad \alpha(\widehat{A}) = \alpha(1);$$
$$\mathcal{M} \nvDash 1 \preccurlyeq A \quad \Rightarrow \quad \alpha(\widehat{A}) = \alpha(0).$$

  - Recall that $\alpha(0) = \varnothing$ and $\alpha(1)$ is $\{\varepsilon\}$ on algebras of languages and δ on algebras of relations.
- For algebras of formal languages: $\widehat{A} = 1 \wedge A$.
- For algebras of binary relations: $\widehat{A} = 1 \,/\, \big(0 \,/\, (0 \,/\, (1 \wedge A))\big)$.

Representation of Horn theories

Lemma

If $\widehat{\cdot}$ is a "classicalisation," then, for the given class of models,

$$1 \preccurlyeq A \vDash 1 \preccurlyeq B \quad \Longleftrightarrow \quad \vDash \widehat{A} \preccurlyeq B.$$

- This is proved by considering two cases, for each model $\mathscr{M}$, whether the formula $1 \preccurlyeq A$ is true or false.
- This lemma is a semantical version of deduction theorem.
- An arbitrary inequality $E \preccurlyeq F$ can be equivalently replaced by $1 \preccurlyeq E \setminus F$.
- Several inequalities (Horn premises) $1 \preccurlyeq A_1$, ..., $1 \preccurlyeq A_n$ are replaced by one: $1 \preccurlyeq (A_1 \wedge \ldots \wedge A_n)$.

Complexity of Equational Theories

- Thus, Horn theories for algebras of formal languages and algebras of binary relations can be reduced to the corresponding equational theories.
- In particular, this holds for the language of $\backslash, /, \backslash\backslash, //, \wedge$, where the corresponding Horn theories are Π_1^1-complete.
  - In the reduction, we extend the language by constants 0 and 1.

Theorem

The equational theory of algebras of formal languages and the equational theory of algebras of binary relations (in the language of RKLs) are Π_1^1-complete.

Conclusion: Open Questions

- **Open question:** complexity of equational theories for the given concrete classes of algebras without constants.
- Another potential source of complexity is the **distributivity law:**

$$(a \vee b) \wedge c \preccurlyeq (a \wedge c) \vee (b \wedge c).$$

- Here we do not know even the complexity of the equational theory for the class of *all* distributive $*$-continuous RKLs.
    - This theory can be axiomatised by a sequent calculus [Kozak 2009], but the upper Π^0_1 bound proof does not work for it.
- A possibly simpler question: for algebras of binary relations, is it sufficient to have only one constant (presumably, 1), in order to prove Π^1_1-hardness?

Some References

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Thank you for your attention!

The $\mathbf{ACT}_\omega$ Calculus

- Sequents of $\mathbf{ACT}_\omega$ [Palka 2007] are expressions of the form $\Pi \to A$, where Π is a sequence of formulae (terms) and A is a formula. The meaning of $A_1, \ldots, A_n \to B$ is $A_1 \cdot \ldots \cdot A_n \preccurlyeq B$, and $1 \preccurlyeq B$ for $n = 0$.
- $\mathbf{ACT}_\omega$ extends the multiplicative-additive Lambek calculus **MALC**:

$$\frac{}{A \to A}\ (\mathrm{id}) \qquad \frac{}{\Gamma, 0, \Delta \to C}\ (0\mathrm{L}) \qquad \frac{\Gamma, \Delta \to C}{\Gamma, 1, \Delta \to C}\ (1\mathrm{L}) \qquad \frac{}{\to 1}\ (1\mathrm{R}) \qquad \frac{\Pi \to A \quad \Gamma, A, \Delta \to C}{\Gamma, \Pi, \Delta \to C}\ (\mathrm{cut})$$

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, \Pi, A \setminus B, \Delta \to C}\ (\setminus\mathrm{L}) \qquad \frac{A, \Pi \to B}{\Pi \to A \setminus B}\ (\setminus\mathrm{R}) \qquad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \cdot B, \Delta \to C}\ (\cdot\mathrm{L})$$

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, B / A, \Pi, \Delta \to C}\ (/\mathrm{L}) \qquad \frac{\Pi, A \to B}{\Pi \to B / A}\ (/\mathrm{R}) \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B}\ (\cdot\mathrm{R})$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C} \quad \frac{\Gamma, B, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C}\ (\wedge\mathrm{L}) \qquad \frac{\Pi \to A \quad \Pi \to B}{\Pi \to A \wedge B}\ (\wedge\mathrm{R})$$

$$\frac{\Gamma, A, \Delta \to C \quad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C}\ (\vee\mathrm{L}) \qquad \frac{\Pi \to A}{\Pi \to A \vee B} \quad \frac{\Pi \to B}{\Pi \to A \vee B}\ (\vee\mathrm{R})$$

The $\mathbf{ACT}_\omega$ Calculus

The rules for Kleene star [Palka 2007] and iterative divisions [Kuznetsov, Ryzhkova 2020; Kuznetsov 2024]:

$$\frac{\big(\Gamma, A^n, \Delta \to C\big)_{n=0}^{\infty}}{\Gamma, A^*, \Delta \to C}\ (*\mathsf{L}_\omega) \qquad \frac{\Pi_1 \to A \quad \dots \quad \Pi_n \to A}{\Pi_1, \dots, \Pi_n \to A^*}\ (*\mathsf{R}_n)$$

$$\frac{\Pi_1 \to A \quad \dots \quad \Pi_n \to A \quad \Gamma, B, \Delta \to C}{\Gamma, \Pi_1, \dots, \Pi_n, A \mathbin{\backslash\backslash} B, \Delta \to C}\ (\backslash\backslash\mathsf{L}_n) \qquad \frac{\big(A^n, \Pi \to B\big)_{n=0}^{\infty}}{\Pi \to A \mathbin{\backslash\backslash} B}\ (\backslash\backslash\mathsf{R}_\omega)$$

$$\frac{\Pi_1 \to A \quad \dots \quad \Pi_n \to A \quad \Gamma, B, \Delta \to C}{\Gamma, B \mathbin{//} A, \Pi_1, \dots, \Pi_n, \Delta \to C}\ (//\mathsf{L}_n) \qquad \frac{\big(\Pi, A^n \to B\big)_{n=0}^{\infty}}{\Pi \to B \mathbin{//} A}\ (//\mathsf{R}_\omega)$$

Π^1_1-Hardness for the Horn Theory

- Kozen [2002] proves Π^1_1-hardness for reasoning from hypotheses (i.e., for the Horn theory) in $*$-continuous Kleene algebras by encoding **well-foundedness** (noetherianity) of a computable binary relation.
- The input of this problem is a Turing machine which computes the relation $R \subseteq \mathbb{N} \times \mathbb{N}$ as follows:
  - if nRm, then the machine reaches $\blacktriangleright x^m t \blacktriangleleft$ from $\blacktriangleright x^n s y^m \blacktriangleleft$;
  - if not nRm, then it reaches $\blacktriangleright r \blacktriangleleft$ from $\blacktriangleright x^n s y^m \blacktriangleleft$.
- The machine is naturally represented by a word rewriting system $\mathcal{S}$.
- We add an "infinitary rewriting rule" $t \Rightarrow sy^*$, to restart the computation after a successful one (when nRm is true).

Π^1_1-Hardness for the Horn Theory

- In the presence of product and Kleene star, $\mathscr{S}$ and the additional infinitary rewriting rule are naturally encoded as a finite set of hypotheses $\mathscr{H}$ over $\mathbf{ACT}_\omega$ [Kozen 2002; Kuznetsov, Speranski 2022].
- Each rewriting rule of the form $u_1 \ldots u_\ell \Rightarrow v_1 \ldots \cdot v_k$ is encoded as $u_1, \ldots, u_\ell \to v_1 \cdot \ldots \cdot v_k$, and $t \Rightarrow sy^*$ is encoded as $t \to s \cdot y^*$.
- Under this encoding, we get the following: there is no infinite R-path starting from zero (noetherianity) if and only if $\mathscr{H} \vdash_{\mathbf{ACT}_\omega} \blacktriangleright, t, \blacktriangleleft \to \blacktriangleright \cdot r \cdot \blacktriangleleft$.
- This establishes Π^1_1-completeness for reasoning from hypotheses in $\mathbf{ACT}_\omega$ (that is, of the Horn theory of all $*$-continuous action lattices).
- However, we need to replace product and Kleene star by division and iterative division.

(Infinitary) Buszkowski's Rules

- Replacing product with division is done via so-called **Buszkowski's rules** [Buszkowski 1982].
- We extend this approach by adding infinitary Buszkowski's rules.
- A Buszkowski's rule is an additional inference rule of the form:

$$\frac{\Pi_1 \to u_1 \quad \ldots \quad \Pi_\ell \to u_\ell \quad \Delta, v_1, \ldots, v_k \to a}{\Pi_1, \ldots, \Pi_\ell, \Delta \to b} \ (\mathrm{B}_{a;b;\vec{u};\vec{v}})$$

- Its infinitary version, for $t \Rightarrow sy^*$, is

$$\frac{\Pi \to t \quad \left(\Delta, s, y^n \to a\right)_{n=0}^{\infty}}{\Pi, \Delta \to b} \ (\mathrm{B}_{a;b;t;s,y^*})$$

- The corresponding **Buszkowski's formulae**:

$$B_{a;b;\vec{u};\vec{v}} = \big(b/(a/(v_1 \ldots v_k))\big)/(u_1 \ldots u_\ell), \qquad B_{a;b;t;s,y^*} = \big(b/((a//y)/s)\big)/t.$$

(Infinitary) Buszkowski's Rules

- Buszkowski's rules (B-rules) and Buszkowski's formulae (B-formulae) are equivalent in the following sense:

Lemma

Let $\mathbf{B}$ *be a finite set of B-rule (maybe infinitary) and let* $\mathscr{H}_{\mathbf{B}}$ *be the set of hypotheses of the form* $\to B$*, where* B *is the corresponding B-formula, for each B-rule from* $\mathbf{B}$*. Then* $\mathscr{H}_{\mathbf{B}} \vdash_{\mathbf{ACT}_\omega} \Pi \to A$ *iff* $\mathbf{ACT}_\omega + \mathbf{B} \vdash \Pi \to A$*, for any sequent* $\Pi \to A$*.*

- Thus, reasoning with B-rules is actually inside the Horn theory, and we need only / and // to simulate it.
- On the other hand, B-rules are more convenient due to cut elimination:

Lemma

For any set of B-rules $\mathbf{B}$*, the* (cut) *rule in* $\mathbf{ACT}_\omega + \mathbf{B}$ *is eliminable.*

Encoding Rewriting Using Buszkowski's Rules

- Using B-rules, a rewriting rule $\mathfrak{p} = (u_1 \dots u_\ell \Rightarrow v_1 \dots v_k)$ from $\mathcal{S}$ is encoded as follows:

$$\frac{\Pi \to \mathbf{e}_\mathfrak{p} \quad \Delta \to \mathbf{f}}{\Pi, \Delta \to \mathbf{b}_\mathfrak{p}}\ (1_\mathfrak{p}) \qquad \frac{\Pi \to w \quad \Delta, w \to \mathbf{b}_\mathfrak{p}}{\Pi, \Delta \to \mathbf{b}_\mathfrak{p}}\ (2_\mathfrak{p}),\ w \in \Sigma \cup \{\blacktriangleleft\}$$

$$\frac{\Pi_1 \to u_1 \quad \dots \quad \Pi_\ell \to u_\ell \quad \Delta, v_1, \dots, v_k \to \mathbf{a}_\mathfrak{p}}{\Pi_1, \dots, \Pi_\ell, \Delta \to \mathbf{b}_\mathfrak{p}}\ (3_\mathfrak{p})$$

$$\frac{\Pi \to w \quad \Delta, w \to \mathbf{a}_\mathfrak{p}}{\Pi, \Delta \to \mathbf{a}_\mathfrak{p}}\ (4_\mathfrak{p}),\ w \in \Sigma \cup \{\blacktriangleright\} \qquad \frac{\Delta, \mathbf{e}_\mathfrak{p} \to \mathbf{a}_\mathfrak{p}}{\Delta \to \mathbf{f}}\ (5_\mathfrak{p})$$

- The simulation of $\mathfrak{p}$ in $\mathbf{ACT}_\omega + \mathbf{B}$ is as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Delta_1, v_1, \dots, v_k, \Delta_2 \to \mathbf{f}}{\mathbf{e}_\mathfrak{p}, \Delta_1, v_1, \dots, v_k, \Delta_2 \to \mathbf{b}_\mathfrak{p}}\ (1_\mathfrak{p})}{\Delta_2, \mathbf{e}_\mathfrak{p}, \Delta_1, v_1, \dots, v_k \to \mathbf{b}_\mathfrak{p}}\ (2_\mathfrak{p})}{u_1, \dots, u_\ell, \Delta_2, \mathbf{e}_\mathfrak{p}, \Delta_1 \to \mathbf{a}_\mathfrak{p}}\ (3_\mathfrak{p})}{\Delta_1, u_1, \dots, u_\ell, \Delta_2, \mathbf{e}_\mathfrak{p} \to \mathbf{a}_\mathfrak{p}}\ (4_\mathfrak{p})}{\Delta_1, u_1, \dots, u_\ell, \Delta_2 \to \mathbf{f}}\ (5_\mathfrak{p})$$

Encoding Rewriting Using Buszkowski's Rules

- For the infinitary rewriting rule, $t \Rightarrow sy^*$, we construct the same B-rules (1_*), (2_*), (4_*), (5_*), and for (3_*) we take the following infinitary B-rule:

$$\frac{\Pi \to t \quad \left(\Delta, s, y^m \to \mathbf{a}_*\right)_{m=0}^{\infty}}{\Pi, \Delta \to \mathbf{b}_*}\ (3_*)$$

- The set of B-rules (1_*)–(5_*) simulates $t \Rightarrow sy^*$, by deriving $\Delta_1, t, \Delta_2 \to \mathbf{f}$ for an infinite series of sequents $\left(\Delta_1, s, y^m, \Delta_2 \to \mathbf{f}\right)_{m=0}^{\infty}$.
- Finally, we add the following B-rule for termination:

$$\frac{\Pi_1 \to \blacktriangleright \quad \Pi_2 \to r \quad \Pi_3 \to \blacktriangleleft \quad \Delta, \mathbf{f} \to \mathbf{f}}{\Pi_1, \Pi_2, \Pi_3, \Delta \to \mathbf{f}}\ (\mathbf{f})$$

Encoding Rewriting Using Buszkowski's Rules

- The encoding of rewriting using the aforementioned set of B-rules **B** is faithful in the following sense.

Lemma

The Turing machine reaches a configuration Ψ *from a configuration* Φ *whose state is neither* t *nor* r*, if and only if the sequent* $\Phi \to \mathbf{f}$ *is cut-free derivable from the sequent* $\Psi \to \mathbf{f}$ *in* $\mathbf{ACT}_\omega + \mathbf{B}$*.*

Lemma

The relation R *(defined by the Turing machine) has no infinite path starting from* n *(for a given* n*) if and only if the sequent* $\blacktriangleright, x^n, t, \blacktriangleleft \to \mathbf{f}$ *is derivable in* $\mathbf{ACT}_\omega + \mathbf{B}$*.*

- Here the "only if" direction is straightforward, and the "if" one is performed by cut-free proof analysis (cut-free proofs here use only B-rules, not the original rules of $\mathbf{ACT}_\omega$).

Encoding Rewriting Using Buszkowski's Rules

- The second lemma immediately yields Π_1^1-hardness for extensions of $\mathbf{ACT}_\omega$ with finite sets of B-rules.
- By equivalence with B-formulae, we get the corresponding result for reasoning from finite sets of hypotheses in $\mathbf{ACT}_\omega$ (the Horn theory), in the necessary restricted fragment.

Theorem

The fragment of the Horn theory of the class of all $*$-continuous RKLs, in the language of $\backslash, /, \backslash\backslash, //, \wedge$, is Π_1^1-complete.

- Finally, by completeness in the restricted language, this also holds for Horn theories of our concrete classes of models (algebras of languages and algebras of relations).