

Amalgamation in Quasivarieties of Sugihara Algebras

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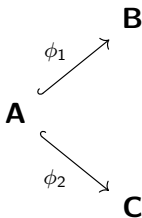
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The amalgamation property

The **amalgamation property** is a fundamental categorical-algebraic property, and it dates back to Schreier's work on group theory.

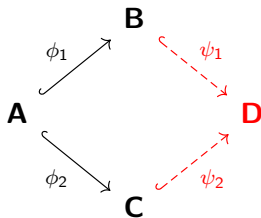
A class K of first-order structures has the amalgamation property (or **AP**) if every span $\langle \phi_1: \mathbf{A} \rightarrow \mathbf{B}, \phi_2: \mathbf{A} \rightarrow \mathbf{C} \rangle$ of structures in K can be completed in K so the following diagram commutes:



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Jónsson's Problem:

Given a first-order theory T , determine whether T has the AP based solely on the shape of its axioms.

Importance to:

- Homogeneous structures, Fraïssé theory
- Interpolation properties

Lots of success for algebraic classes, in particular **varieties**:

- Heyting algebras/IPC, S4-algebras (Maksimova 1977), N3-algebras (Goranko 1985)
- Models of linear logic (F.-Santschi 2024)
- Solution of the Exchange Conjecture (Gil-Férez-Ledda-Tsinakis 2015, Gil-Férez-Jipsen-Metcalfe 2020, Fussner-Galatos 2024&2025, Fussner-Metcalfe-Santschi 2026)
- Solution of Montagna's problem/BL-algebras (Montagna 2006, Aguzzoli-Bianchi 2021&2023, Fussner-Santschi 2025)
- Classification for varieties of Sugihara monoids (Marchioni-Metcalfe 2012)

Today: **Arbitrary quasivarieties** of Sugihara algebras

Sugihara algebras

Let \cdot be meet with respect to the following non-standard order on \mathbb{Z} :

$$\dots - 3 < 3 < -2 < 2 < -1 < 1 < 0.$$

Define an algebra $\mathbf{Z} = \langle \mathbb{Z}, \wedge, \vee, \cdot, \rightarrow, - \rangle$. A **Sugihara algebra** is an algebra in the variety $\text{HSP}(\mathbf{Z})$ (just 0-free subreducts of a **Sugihara monoids**).

Can leave out zero to obtain the algebra \mathbf{E} , or cut-off either \mathbf{Z} or \mathbf{E} to obtain finite Sugihara algebras \mathbf{Z}_n .

Lattice of subvarieties of $\text{SA} = \text{HSP}(\mathbf{Z})$:

$$V(\mathbf{Z}_1) \subseteq V(\mathbf{Z}_2) \subseteq V(\mathbf{Z}_3) \subseteq \dots \subseteq V(\mathbf{E}) = V(\mathbf{Z}).$$

Lattice of **quasivarieties**: Absolutely wild. Not congruence distributive, don't have relative **congruence extension property**

Theorem 1:

There are exactly five quasivarieties of Sugihara algebras with the AP.

Theorem 2:

For any quasivariety of Sugihara algebras, the AP implies the CEP.

Theorem 3:

The problem of determining whether a finitely based quasivariety of Sugihara algebras has the AP is effectively decidable.

We'll sketch the proof of [Theorem 1](#).

Some closure conditions

Lemma 1:

Let Q be any quasivariety of Sugihara algebras properly containing $V(\mathbf{Z}_2)$. Then either $\mathbf{Z}_2 \times \mathbf{Z}_3 \in Q$ or $\mathbf{Z}_2 \times \mathbf{Z}_4 \in Q$.

Lemma 2:

If Q is any quasivariety of Sugihara algebras with the AP and $\mathbf{Z}_2 \times \mathbf{Z}_3 \in Q$, then $\mathbf{Z}_3 \in Q$.

Lemma 2:

If Q is any quasivariety of Sugihara algebras with the AP and $\mathbf{Z}_2 \times \mathbf{Z}_4 \in Q$, then $\mathbf{E} \in Q$.

Lemma 3:

If Q is any quasivariety of Sugihara algebras with the AP and $\mathbf{Z}_3, \mathbf{Z}_4 \in Q$. Then $Q = SA$.

Theorem 4:

There are exactly five extensions (not necessarily axiomatic) of the relevant logic **RM** with the Maehara interpolation property.

In fact, all of the quasivarieties/logics with AP/MIP are finitely based.

The most interesting of these is $Q(\mathbf{E})$, which is axiomatized relative to Sugihara algebras by the rules

$$\alpha, \neg\alpha \vdash \beta, \quad \alpha, \neg\alpha \vee \beta \vdash \beta.$$

Theorem (Kearnes, 1989):

Let V be any residually small (up to isomorphism, only a set of subdirectly irreducibles), congruence modular variety. Then if V has the AP, it follows that V has the CEP.

Known also to be true of any variety of semigroups.

Here a similar result holds far beyond that setting.

Thank you!

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