String Diagrams and Homotopies in Simplicial Objects

RAMiCS 2026

Aloÿs Dufour
joint work with Pierre-Louis Curien

LIPN, Université Sorbonne Paris Nord

April 8, 2026

Context

Contractible simplicial objects[1]

- Several non-equivalent notions of contractibility in simplicial objects

[1] Barr, M., Kennison, J. F., and Raphael, R. "Contractible simplicial objects". English. In: *Commentationes Mathematicæ Universitatis Carolinæ* 60 (4 2019). ISSN: 0010-2628. DOI: https://doi.org/10.14712/1213-7243.2019.023. URL: http://hdl.handle.net/10338.dmlcz/147968.

Context

Contractible simplicial objects[1]

- Several non-equivalent notions of contractibility in simplicial objects
- Leading to the description of when there are equivalent

[1] Barr, M., Kennison, J. F., and Raphael, R., "Contractible simplicial objects".

Context

Contractible simplicial objects[1]

- Several non-equivalent notions of contractibility in simplicial objects
- Leading to the description of when there are equivalent
- In fact different notions of homotopies

[1] Barr, M., Kennison, J. F., and Raphael, R., "Contractible simplicial objects".

Context

Contractible simplicial objects[1]

- Several non-equivalent notions of contractibility in simplicial objects
- Leading to the description of when there are equivalent
- In fact different notions of homotopies
- Very combinatorial proofs

[1] Barr, M., Kennison, J. F., and Raphael, R., "Contractible simplicial objects".

Simplicies and simplicial objects

Definition

The category Δ is given by:

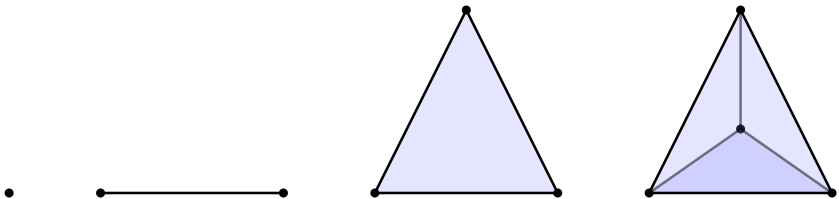
- Objects: non-empty ordinals $[n] := \{0 < 1 < \cdots < n\}$,
- Morphisms: non-decreasing maps.

Simplicies and simplicial objects

Definition

The category Δ is given by:

- Objects: non-empty ordinals $[n] := \{0 < 1 < \cdots < n\}$,
- Morphisms: non-decreasing maps.

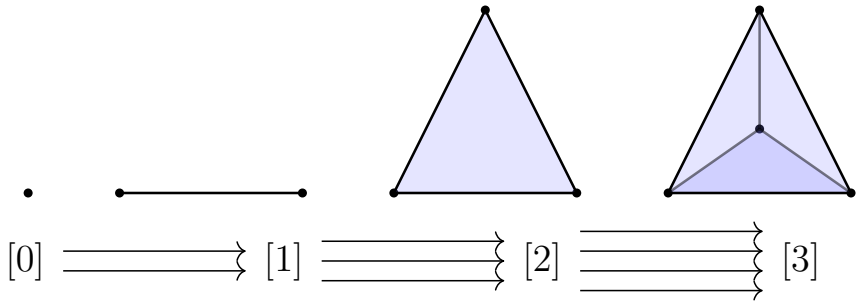


Simplicies and simplicial objects

Definition

The category Δ is given by:

- Objects: non-empty ordinals $[n] := \{0 < 1 < \cdots < n\}$,
- Morphisms: non-decreasing maps.



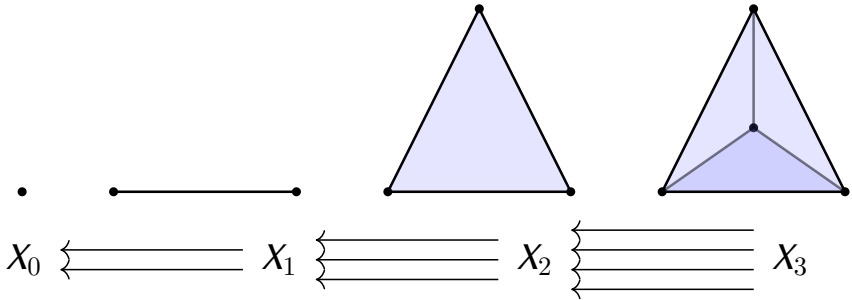
Simplicies and simplicial objects

Definition

The category Δ is given by:

- Objects: non-empty ordinals $[n] := \{0 < 1 < \cdots < n\}$,
- Morphisms: non-decreasing maps.

A *simplicial object* X in a category $\mathcal{C}$ is $X : \Delta^{\mathrm{op}} \to \mathcal{C}$.



Classical presentation of Δ

$n, m \in \mathbb{N}$,

- Non-decreasing injections are generated by $\delta^i : [n-1] \to [n]$, for $i \in [\![0, n]\!]$ and

$$\delta^i(j) := j \text{ if } j < i \quad \text{and} \quad \delta^i(j) := j+1 \text{ if } j \geqslant i;$$

Classical presentation of Δ

$n, m \in \mathbb{N}$,

- Non-decreasing injections are generated by $\delta^i : [n-1] \to [n]$, for $i \in [\![0, n]\!]$ and

$$\delta^i(j) := j \text{ if } j < i \quad \text{and} \quad \delta^i(j) := j+1 \text{ if } j \geqslant i;$$

- Non-decreasing surjections are generated by $\sigma^j : [n] \to [n-1]$, for $j \in [\![0, n-1]\!]$ and

$$\sigma^j(k) := k \text{ if } k \leqslant j \quad \text{and} \quad \sigma^j(k) := k-1 \text{ if } k \geqslant j+1;$$

Classical presentation of Δ

$n, m \in \mathbb{N}$,

- Non-decreasing injections are generated by $\delta^i : [n-1] \to [n]$, for $i \in [\![0, n]\!]$ and

$$\delta^i(j) := j \text{ if } j < i \quad \text{and} \quad \delta^i(j) := j+1 \text{ if } j \geqslant i;$$

- Non-decreasing surjections are generated by $\sigma^j : [n] \to [n-1]$, for $j \in [\![0, n-1]\!]$ and

$$\sigma^j(k) := k \text{ if } k \leqslant j \quad \text{and} \quad \sigma^j(k) := k-1 \text{ if } k \geqslant j+1;$$

- *Co-simplicial* relations are given by:

$$\begin{cases} \delta^j\delta^i = \delta^i\delta^{j-1}, & \text{if } i < j \\ \sigma^j\delta^i = \delta^i\sigma^{j-1}, & \text{if } i < j \\ \sigma^j\delta^i = \mathrm{Id}, & \text{if } i = j, j+1 \\ \sigma^j\delta^i = \delta^{i-1}\sigma^j, & \text{if } i > j+1 \\ \sigma^j\sigma^i = \sigma^{i-1}\sigma^j, & \text{if } i > j. \end{cases}$$

Definition

Let $f, g : \mathrm{Set}^{\Delta^{\mathrm{op}}}[X, Y]$, a *cylindrical homotopy* η between f and g is a morphism $\eta : X \times \Delta[1] \to Y$ such that

$$\begin{array}{ccccc} X \cong \Delta[0] \times X & \xrightarrow{d_0 \times \mathrm{Id}} & \Delta[1] \times X & \xleftarrow{d_1 \times \mathrm{Id}} & \Delta[0] \times X \cong X \\ & \searrow^{f} & \downarrow{\scriptstyle \eta} & \swarrow_{g} & \\ & & Y & & \end{array}$$

commutes.

Definition

Let $f, g : \mathrm{Set}^{\Delta^{\mathrm{op}}}[X, Y]$, a *combinatorial homotopy* between f and g is given by a family of morphisms $(h_{i,n} : X_n \to Y_{n+1})_{i:[\![0,n]\!]}$ in Set, such that

- $d_0 h_0 = f_n$,
- $d_{n+1} h_n = g_n$,
- $d_i h_j = \begin{cases} h_{j-1} d_i, & i < j \\ d_i h_{i-1}, & i = j \neq 0 \\ h_j d_{i-1}, & i > j+1 \end{cases}$,
- $s_i h_j = \begin{cases} h_{j+1} s_i, & i \leqslant j \\ h_j s_{i-1}, & i > j \end{cases}$.

Definition

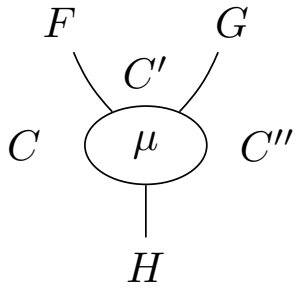
Let $f, g : \mathrm{Set}^{\Delta^{\mathrm{op}}}[X, Y]$, a *combinatorial homotopy* between f and g is given by a family of morphisms $(h_{i,n} : X_n \to Y_{n+1})_{i:[\![0,n]\!]}$ in Set, such that

- $d_0 h_0 = f_n$,
- $d_{n+1} h_n = g_n$,
- $d_i h_j = \begin{cases} h_{j-1} d_i, & i < j \\ d_i h_{i-1}, & i = j \neq 0 \\ h_j d_{i-1}, & i > j+1 \end{cases}$,
- $s_i h_j = \begin{cases} h_{j+1} s_i, & i \leqslant j \\ h_j s_{i-1}, & i > j \end{cases}$.

Link with cylindrical homotopy?

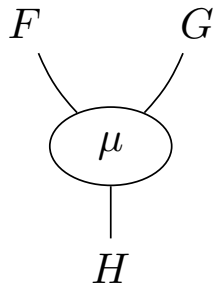
String diagrams

Take a natural transformation $\mu : GF \to H$, between functors $F : C \to C'$, $G : C' \to C''$ and $H : C \to C''$



String diagrams

Take a natural transformation $\mu : GF \to H$, between functors $F : C \to C'$, $G : C' \to C''$ and $H : C \to C''$

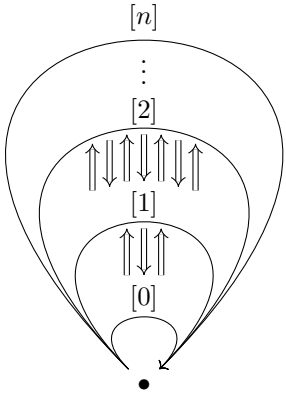


The trick: Δ as 2-category

One object: $\bullet$,

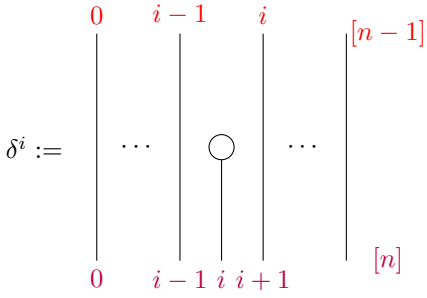
1-morphisms: $[n] := [0]^n$, $n \in \mathbb{N}$,

2-morphisms: between $[n]$ and $[m]$ being $\Delta([n], [m])$.



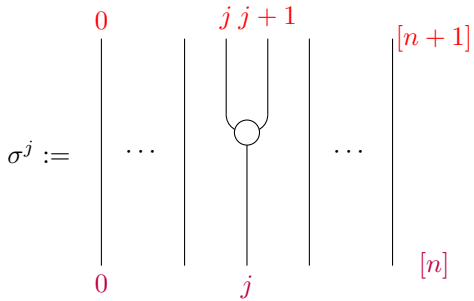
Δ WITH STRING DIAGRAMS

Co-degeneracies



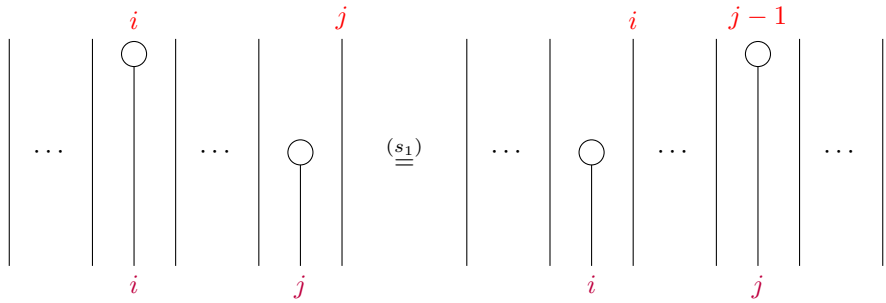
Δ WITH STRING DIAGRAMS

Co-faces



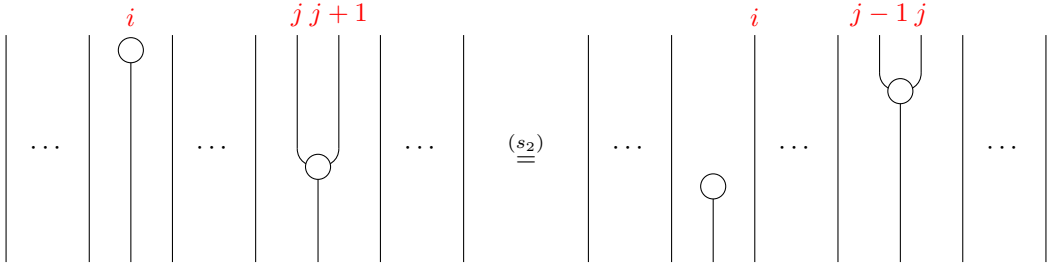
Δ WITH STRING DIAGRAMS

$\delta^j\delta^i = \delta^i\delta^{j-1}$ with $i < j$



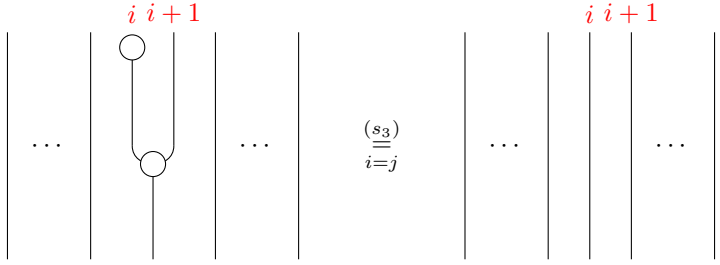
Δ WITH STRING DIAGRAMS

$\sigma^j \delta^i = \delta^i \sigma^{j-1}$, with $i < j$



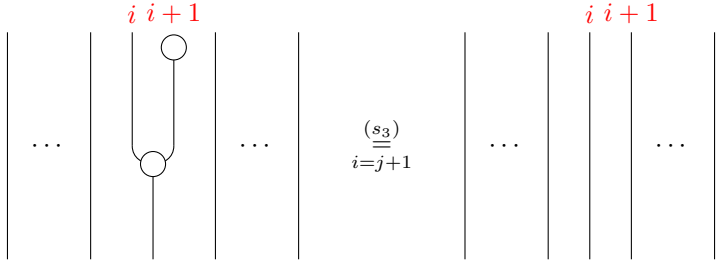
Δ WITH STRING DIAGRAMS

$\sigma^j \delta^i = \mathrm{Id}$, for $i = j, j+1$



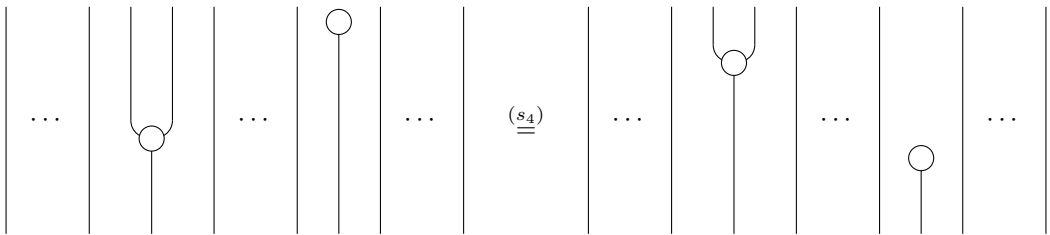
Δ WITH STRING DIAGRAMS

$\sigma^j \delta^i = \mathrm{Id}$, for $i = j, j+1$



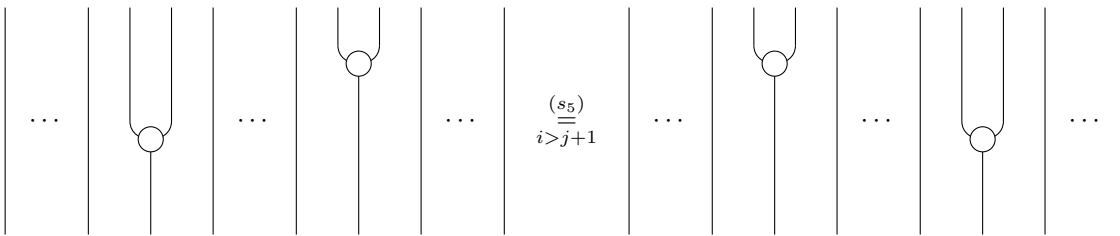
Δ WITH STRING DIAGRAMS

$\sigma^j \delta^i = \delta^{i-1} \sigma^j$, with $i > j+1$



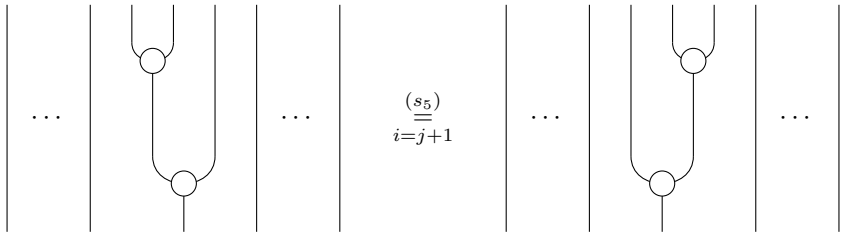
Δ WITH STRING DIAGRAMS

$\sigma^j\sigma^i = \sigma^{i-1}\sigma^j$, with $i > j$



Δ WITH STRING DIAGRAMS

$\sigma^j \sigma^i = \sigma^{i-1} \sigma^j$, with $i > j$



Cylindrical and combinatorial homotopies are equavalent

1. for $i < j$, $d_i\hbar_j \equiv d_i d_{j+1} h_j = d_j d_i h_j = d_j h_{j-1} d_i \equiv \hbar_{j-1} d_i$ using (s1);
2. for $i > j+1$, $d_i\hbar_j \equiv d_i d_{j+1} h_j = d_{j+1} d_{i+1} h_j = d_{j+1} h_j d_i \equiv \hbar_j d_i$;
3. for $i = j \neq 0$, $d_i\hbar_i \equiv d_i d_{i+1} h_i = d_i d_i h_i = d_i d_i h_{i-1} = d_i d_{i+1} h_{i-1} = d_i h_{i-1} d_i \equiv \hbar_{i-1} d_i$ using (s4);
4. for $i = j = 0$, $d_0\hbar_0 \equiv d_0 d_1 h_0 = d_0 d_0 h_0 = d_0 f_n = f_n d_0 \equiv \hbar_{-1} d_0$ using (s1);
5. for $i = j$, $s_i\hbar_i \equiv s_i d_{i+1} h_i = d_{i+1} s_{i+1} h_i = d_{i+1} h_i s_i \equiv \hbar_i s_i$ using (s2);
6. for $i < j$, $s_i\hbar_j \equiv s_i d_{j+1} h_j = d_{j+2} s_i h_j = d_{j+2} h_{j+1} s_i \equiv \hbar_{j+1} s_i$ using (s4);
7. for $i > j$, $s_i\hbar_j \equiv s_i d_{j+1} h_j = d_{j+1} s_{i+1} h_j = d_{j+1} h_j s_i \equiv \hbar_j s_i$ using (s4).

Work

- link with *reduced* homotopy
- Contractibility as homotopical to constant simplicial sets
- Contractibility as augmentations and extra-degeneracies