Deductive Interpolation from Horn Semantics

Peter Arndt, Hugo Mariano and Darllan Pinto

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$\boxed{\text{Logics}}$

Logic: Σ signature $\quad \rightsquigarrow \quad Fm(X)$
X variables

$Th \subseteq \mathcal{P}(Fm(X))$ theory lattice, closed under $\cap$

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Logic: Σ signature, X variables $\rightsquigarrow \mathrm{Fm}(X)$

$\mathrm{Th} \subseteq \mathcal{P}(\mathrm{Fm}(X))$ theory lattice, closed under $\cap$

substitution invariance:

$$\begin{tikzcd}
\mathrm{Fm}(X) & \mathrm{Th} \arrow[r, hook] \arrow[d, "\sigma^{-1}"'] & \mathcal{P}(\mathrm{Fm}(X)) \arrow[d, "\sigma^{-1}"] \\
\mathrm{Fm}(X) \arrow[u, "\sigma"] & \mathrm{Th} \arrow[r, hook] & \mathcal{P}(\mathrm{Fm}(X))
\end{tikzcd}$$

Framework: Filter pairs

$$L \xrightarrow{h} \mathcal{P}(Fm(X))$$

complete lattice

map preserving $\bigwedge$

$Im(h) =: Th$

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$$\begin{tikzcd} Fm(X) \\ Fm(X) \arrow[u, "\sigma"] \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, "h"] \arrow[d, "\sigma^*"'] & \mathcal{P}(Fm(X)) \arrow[d, "\sigma^{-1}"] \\ L \arrow[r, "h"] & \mathcal{P}(Fm(X)) \end{tikzcd}$$

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\end{tikzcd}$$

Def: A filter pair is a pair (G, h) where

$G: \mathcal{C} \longrightarrow Lat$ functor, $\quad h: G \longrightarrow \mathcal{P}(-)$

$\mathcal{C} \subseteq \Sigma\text{-}Str^{op}$

h: natural transformation preserving $\wedge$

Framework: Filter pairs

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[r, phantom, shift right=4, "\perp"] & \mathcal{P}(Fm(X)) \arrow[l, bend left] \end{tikzcd}$$

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$Im(h) =: Th$

$$\begin{tikzcd} Fm(X) & & L \arrow[r, "h"] \arrow[d, "\sigma^*"'] & \mathcal{P}(Fm(X)) \arrow[d, "\sigma^{-1}"] \\ Fm(X) \arrow[u, "\sigma"] & & L \arrow[r, "h"] & \mathcal{P}(Fm(X)) \end{tikzcd}$$

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$G : \mathcal{E} \longrightarrow \mathrm{Lat}$ functor, $\quad h : G \longrightarrow \wp(-)$

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Example: Congruence filter pairs $\quad K$ quasivariety

$G := \mathrm{Co}_K : \Sigma\text{-Str}^{op} \longrightarrow \mathrm{Lat}$

$$\begin{tikzcd} A \arrow[r, mapsto] & \mathrm{Co}_K(A) \arrow[d, "f^*"] \\ B \arrow[u, "f"] \arrow[r, mapsto] & \mathrm{Co}_K(B) \end{tikzcd}$$

$$\mathrm{Co}_K(A) \xrightarrow{h_A} \wp(A)$$

$$\Theta \longmapsto \{ a \mid \delta_i(a) = \varepsilon_i(a) \text{ in } A/_\Theta \ \forall i \in I \}$$

$(\delta_i(x), \varepsilon_i(x))$ pairs of unary terms

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$(\delta_i(x), \varepsilon_i(x))$ pairs of unary terms

Presentation by congruence filter pair $\triangleq$ Alg. semantics;

$$\Gamma \vdash \varphi \iff \{ \delta_i(\gamma) = \varepsilon_i(\gamma) \mid \gamma \in \Gamma, \ i \in I \} \models_K \delta_i(\varphi) = \varepsilon_i(\varphi)$$

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\end{tikzcd}$$

$$\begin{tikzcd}
Co_K(A) \arrow[r, shift left, "h_A"] & \mathcal{P}(A) \arrow[l, shift left, "\Xi_A"]
\end{tikzcd}$$

$$\theta \longmapsto \{ a \mid \delta_i(a) = \varepsilon_i(a) \text{ in } A/\theta \ \forall i \in I \}$$

$(\delta_i(x), \varepsilon_i(x))$ pairs of unary terms

$$Co_K\left(\{ (\delta_i(\gamma), \varepsilon_i(\gamma)) \mid \substack{\gamma \in \Gamma \\ i \in I} \} \right) \longleftarrow\!\!| \ \Gamma$$

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Example: Horn filter pairs

$\Sigma^+ = \Sigma \cup \{\text{relation symbols}\}$

K class of models of a Horn theory $\mathbb{T}$

$\tilde{\Sigma} \subseteq \Sigma^+$ subsignature

$$G = \mathrm{At}_K^{\tilde{\Sigma}} : \Sigma\text{-}\mathrm{Str}^{op} \longrightarrow \mathrm{Lat}$$

$$A \longmapsto \{\mathbb{T}\text{-closed atomic } \tilde{\Sigma}\text{-theories extending the theory of } A\}$$

$$\mathrm{At}_K^{\tilde{\Sigma}}(A) \longrightarrow \mathcal{P}(A)$$

$$T \longmapsto \{a \mid \psi_i(a) \in T \ \ \forall i \in I\}$$

ψ_i some unary atomic Σ^+-formulas

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ψ_i some unary atomic Σ^+-formulas

E.g. $\Sigma^+ = \Sigma \cup \{\leq\}$ or $\Sigma^+ = \Sigma \cup \{R\}$ unary

$\psi(a) = \top \leq a$ $\qquad$ $\psi(a) = R(a)$

Deductive Interpolation Property (DIP):

$\Gamma \vdash \varphi$, then $\Gamma \vdash \psi_i$, $\{\psi_i \mid i \in I\} \vdash \varphi$

$Var(\Gamma) = X \quad Var(\varphi) = Y \qquad Var(\{\psi_i \mid i \in I\}) \subseteq X \cap Y$

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Flat theory amalgamation [Czelakowski-Piggozzi]:

$$\begin{tikzcd}
& Fm(X \cap Y) \arrow[r, hook] \arrow[d, hook] & Fm(Y) \arrow[d, hook] \\
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T \arrow[ur, phantom, "\subseteq" description] & &
\end{tikzcd}$$

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$$
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& Fm(X) \arrow[r, hook] & Fm(X\cup Y) \arrow[dr, phantom, "\supseteq"] & \\
T \arrow[ur, phantom, "\subseteq"] & & & \exists R \arrow[uuu, bend right] \arrow[lll, bend left]
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T & & & & \exists R \arrow[llll, mapsto, bend left] \arrow[uuu, mapsto, bend right]
\end{tikzcd}$$

Flat theory amalg.

$\Downarrow$

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T & & & \exists R \arrow[lll, bend left, maps to] \arrow[uuu, maps to]
\end{tikzcd}$$

Flat theory amalg.

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DIP

Theory lifting property

$$
\begin{tikzcd}
G(Fm(X\cap Y)) \arrow[dr, "h_{X\cap Y}"] & & G(Fm(Y)) \arrow[ll] \arrow[dr, "h_Y"] & \\
& \mathcal{P}(Fm(X\cap Y)) & & \mathcal{P}(Fm(Y)) \arrow[ll] \\
G(Fm(X)) \arrow[uu] \arrow[dr, "h_X"] & & T|_{X\cap Y} \arrow[r, dashed, mapsto] & T' \\
& \mathcal{P}(Fm(X)) \arrow[uu] & T \arrow[u, mapsto] &
\end{tikzcd}
$$

Theory lifting property

$$
\begin{tikzcd}
\Theta_{T|_{X\cap Y}} & & & & \Theta_{T'} \arrow[llll, mapsto] & \\
& G(\mathrm{Fm}(X\cap Y)) \arrow[dr, "h_{X\cap Y}"] & & & G(\mathrm{Fm}(Y)) \arrow[lll] \arrow[dr, "h_Y"] & \\
& & \mathcal{P}(\mathrm{Fm}(X\cap Y)) & & & \mathcal{P}(\mathrm{Fm}(Y)) \arrow[lll] \\
& & & T|_{X\cap Y} \arrow[rr, dashed, mapsto] & & T' \\
\Theta_T \arrow[uuuu, mapsto] & G(\mathrm{Fm}(X)) \arrow[uuu] \arrow[dr, "h_X"] & & & & \\
& & \mathcal{P}(\mathrm{Fm}(X)) \arrow[uuu] & T \arrow[uu, mapsto] & &
\end{tikzcd}
$$

Theorem: For a congruence filter pair (Co_K, h) with the theory lifting property:

$$\text{Amalgamation in } K \implies \text{DIP}$$

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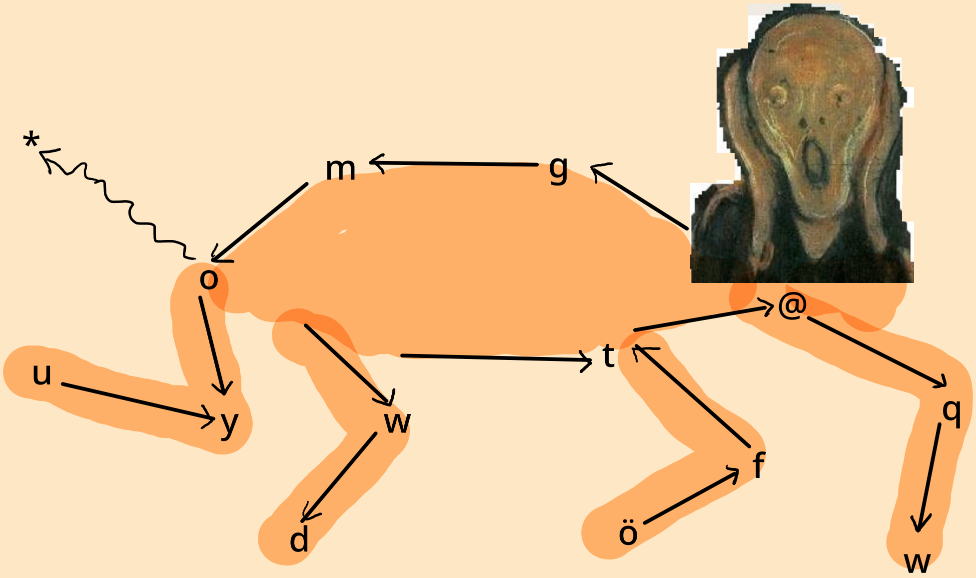

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$$\begin{tikzcd}
\Theta_{T|_{X\cap Y}} & & & \Theta_{T'} \arrow[lll, mapsto] & & \\
& G(Fm(X\cap Y)) \arrow[dr, "h_{X\cap Y}"] & & G(Fm(Y)) \arrow[ll] \arrow[dr, "h_Y"] & & \\
& & \mathcal{P}(Fm(X\cap Y)) & & \mathcal{P}(Fm(Y)) \arrow[ll] & \\
& & & T|_{X\cap Y} \arrow[rr, dashed] & & T' \\
\Theta_T \arrow[uuuu, mapsto] & G(Fm(X)) \arrow[uuu] \arrow[dr, "h_X"] & & & & \\
& & \mathcal{P}(Fm(X)) \arrow[uuu] & & & \\
& & & T \arrow[uuu, mapsto] & &
\end{tikzcd}$$

$$\begin{tikzcd}
Fm(X\cap Y)/\Theta_{X\cap Y} \arrow[r, hook] \arrow[d, hook] & Fm(Y)/\Theta_{T'} \\
Fm(X)/\Theta_T &
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Theorem: For a congruence filter pair (Co_K, h) with the theory lifting property:

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& G(Fm(X\cap Y)) \arrow[dr, "h_{X\cap Y}"] & & G(Fm(Y)) \arrow[ll] \arrow[dr, "h_Y"] & & \\
& & \mathcal{P}(Fm(X\cap Y)) & & \mathcal{P}(Fm(Y)) \arrow[ll] & \\
& & & T|_{X\cap Y} \arrow[rr, dashed, mapsto] & & T' \\
\Theta_T \arrow[uuuu, mapsto] & G(Fm(X)) \arrow[uuu] \arrow[dr, "h_X"] & & & & \\
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& & \mathcal{P}(Fm(X\cap Y)) & & \mathcal{P}(Fm(Y)) \arrow[ll] & \\
& & & T|_{X\cap Y} \arrow[rr, dashed] & & T' \\
\Theta_T \arrow[uuuu, mapsto] & G(Fm(X)) \arrow[uuu] \arrow[dr, "h_X"] & & & & \\
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\end{tikzcd}$$

$$\begin{tikzcd}
Fm(X\cap Y)/\Theta_{X\cap Y} \arrow[r, hook] \arrow[d, hook] & Fm(Y)/\Theta_{T'} \arrow[d, hook] \arrow[ddr, hook, bend left] & \\
Fm(X)/\Theta_T \arrow[r, hook] \arrow[drr, hook, bend right] & P \arrow[dr, dotted] & \\
& & A
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 & & & T|_{X\cap Y} \arrow[rr, dashed, mapsto] & & T' \\
\Theta_T \arrow[uuuu, mapsto] & G(Fm(X)) \arrow[uuu] \arrow[dr, "h_X"] & & & & \\
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& & & T|_{X\cap Y} \arrow[r, dashed] & T' \\
\Theta_T \arrow[uuuu, mapsto] & G(Fm(X)) \arrow[uuu] \arrow[dr, "h_X"] \\
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\end{tikzcd}$$

$$P = Fm(X\cup Y)/\Theta_T \vee \Theta_{T'}$$

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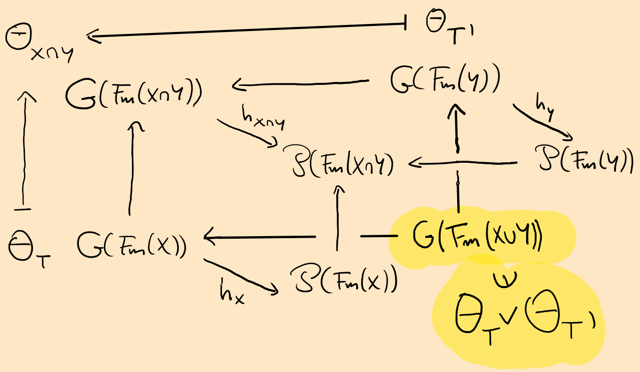

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$$\begin{tikzcd}
\Theta_{X\cap Y} & & & \Theta_{T'} \arrow[lll, mapsto] & \\
 & G(Fm(X\cap Y)) \arrow[dr, "h_{X\cap Y}"] & & G(Fm(Y)) \arrow[ll] \arrow[dr, "h_Y"] & \\
 & & \mathcal{P}(Fm(X\cap Y)) & & \mathcal{P}(Fm(Y)) \arrow[ll] \\
\Theta_T \arrow[uuu, mapsto] & G(Fm(X)) \arrow[uu] \arrow[dr, "h_X"'] & & G(Fm(X\cup Y)) \arrow[ll] \arrow[uu] \arrow[d, phantom, sloped, "\ni"] & \\
 & & \mathcal{P}(Fm(X)) \arrow[uu] & \Theta_T \vee \Theta_{T'} \arrow[uuuu, mapsto, bend right=60] \arrow[ulll, mapsto, bend left] &
\end{tikzcd}$$

$$\begin{tikzcd}
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& & \mathcal{P}(Fm(X)) \arrow[uu] & & \mathcal{P}(Fm(X\cup Y)) \arrow[ll] \arrow[uu] \\
& & \Theta_T \vee \Theta_{T'} \arrow[u, phantom, sloped, "\in"] \arrow[rr, mapsto] & & R \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

flat theory amalgamation

$$\begin{tikzcd}
Fm(X\cap Y)/\Theta_{X\cap Y} \arrow[r, hook] \arrow[d, hook] & Fm(Y)/\Theta_{T'} \arrow[d, hook] \\
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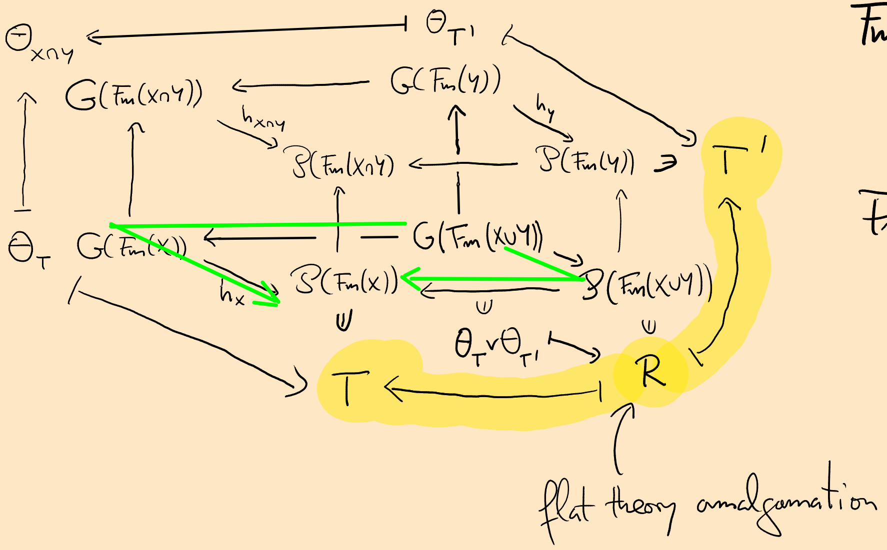
Theorem: For a congruence filter pair (Co_K, h) with the theory lifting property:

$$\text{Amalgamation in } K \implies \text{DIP}$$



$$\begin{tikzcd}
Fm(X\cap Y)/\Theta_{X\cap Y} \arrow[r, hook] \arrow[d, hook] & Fm(Y)/\Theta_{T'} \arrow[d, hook] \\
Fm(X)/\Theta_T \arrow[r, hook] & P
\end{tikzcd}$$

$$P = Fm(X\cup Y)/\Theta_T \vee \Theta_{T'}$$

Theorem: For a ~~congruence~~ Horn filter pair (Co_K, h) with the theory lifting property:

$$\text{Amalgamation in } K \implies \text{DIP}$$

$$\begin{tikzcd}
\overset{+I|_{X\cap Y}}{\Theta_{X\cap Y}} & & & \overset{+I'}{\Theta_{T'}} \arrow[lll, mapsto] \arrow[ddrr, mapsto] & & \\
& G(Fm(X\cap Y)) \arrow[dr, "h_{X\cap Y}"] & & G(Fm(Y)) \arrow[ll] \arrow[dr, "h_Y"] & & \\
& & \mathcal{P}(Fm(X\cap Y)) & & \mathcal{P}(Fm(Y)) \arrow[ll] & T' \\
\underset{+I}{\Theta_T} \arrow[uuu, mapsto] \arrow[ddrr, mapsto] & G(Fm(X)) \arrow[uu] \arrow[dr, "h_X"] & & & & \\
& & \mathcal{P}(Fm(X)) \arrow[uu, "T|_{X\cap Y}"'] & & & \\
& & T & & &
\end{tikzcd}$$

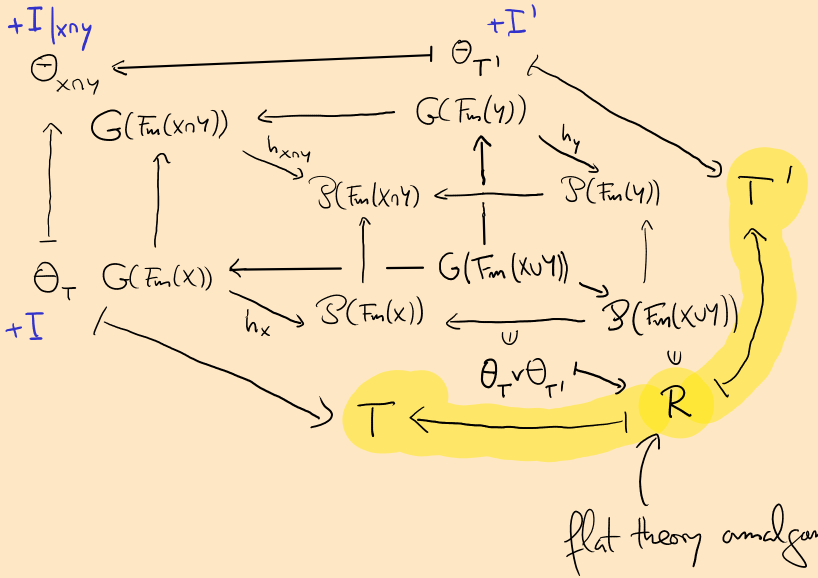
$$Fm(X\cap Y)/\Theta_{X\cap Y} \quad +I|_{X\cap Y}$$

$$Fm(Y)/\Theta_{T'} \quad +I'$$

$$Fm(X)/\Theta_T \quad +I$$

Theorem: For a ~~congruence~~ Horn filter pair (Co_K, h) with the theory lifting property:

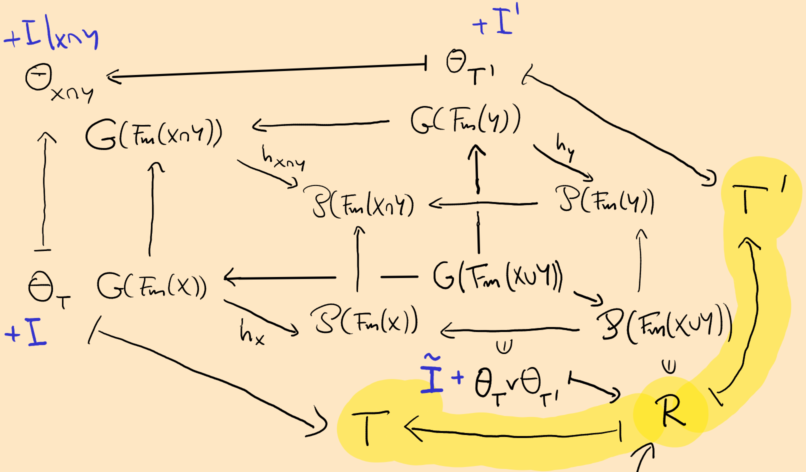
Amalgamation in K $\Longrightarrow$ DIP



$$\begin{tikzcd}
\mathrm{Fm}(X\cap Y)/\Theta_{X\cap Y} \; {+I|_{X\cap Y}} \arrow[r, tail, "\text{atomic embeddings}"] \arrow[d, tail] & \mathrm{Fm}(Y)/\Theta_{T'} \; {+I'} \\
\mathrm{Fm}(X)/\Theta_T \; {+I} &
\end{tikzcd}$$

Theorem: For a ~~congruence~~ Horn filter pair (Co_K, h) with the theory lifting property:

Atomic Amalgamation in K $\Longrightarrow$ DIP



flat theory amalgamation

atomic embeddings

$$
\begin{tikzcd}
\underset{+I|_{X\cap Y}}{Fm(X\cap Y)/\Theta_{X\cap Y}} \arrow[r, tail] \arrow[d, tail] & \underset{+I'}{Fm(Y)/\Theta_{T'}} \arrow[d, tail] \\
\underset{+I}{Fm(X)/\Theta_T} \arrow[r, tail] & P
\end{tikzcd}
$$

$$P = Fm(X\cup Y)/\Theta_T \vee \Theta_{T'} \quad + \tilde{I}$$

Remaining question:

When does a filter pair have the theory lifting property?

Sufficient: Ξ is natural

$$\begin{tikzcd}
G(Fm(X\cap Y)) & & G(Fm(Y)) \arrow[ll] & \\
& \wp(Fm(X\cap Y)) \arrow[ul, "\Xi_{X\cap Y}"'] & & \wp(Fm(Y)) \arrow[ll] \arrow[ul, "\Xi_Y"'] \\
G(Fm(X)) \arrow[uu] & & & \\
& \wp(Fm(X)) \arrow[uu] \arrow[ul, "\Xi_X"] & &
\end{tikzcd}$$

Remaining question:

When does a filter pair have the theory lifting property?

Sufficient: Ξ is natural

$$
\begin{tikzcd}
G(\mathrm{Fm}(X \cap Y)) & & G(\mathrm{Fm}(Y)) \arrow[ll] & \\
 & \wp(\mathrm{Fm}(X \cap Y)) \arrow[ul, "\Xi_{X \cap Y}"'] & & \wp(\mathrm{Fm}(Y)) \arrow[ul, "\Xi_Y"'] \arrow[ll] \\
G(\mathrm{Fm}(X)) \arrow[uu] & & & \\
 & \wp(\mathrm{Fm}(X)) \arrow[ul, dash, "\Xi_X"] \arrow[uu] & &
\end{tikzcd}
$$

Sufficient for that: h is injective

Remaining question:

When does a filter pair have the theory lifting property?

Sufficient: Ξ is natural

$$\begin{tikzcd}
G(Fm(X \cap Y)) & & G(Fm(Y)) \arrow[ll] & \\
& \mathfrak{F}(Fm(X \cap Y)) \arrow[ul, "\Xi_{X \cap Y}"] & & \mathfrak{F}(Fm(Y)) \arrow[ll] \arrow[ul, "\Xi_Y"'] \\
G(Fm(X)) \arrow[uu] & & & \\
& \mathfrak{F}(Fm(X)) \arrow[ul, dash, "\Xi_X"] \arrow[uu] & &
\end{tikzcd}$$

Sufficient for that: h is injective

for congruence filter pairs $\longrightarrow$ $\Updownarrow$

(G, h) presents an <u>equivalent</u> alg. semantics

Remaining question:

When does a filter pair have the theory lifting property?

Sufficient: Ξ is natural

$$\begin{tikzcd}
G(Fm(X\cap Y)) & & G(Fm(Y)) \arrow[ll] & \\
& \wp(Fm(X\cap Y)) \arrow[ul, "\Xi_{X\cap Y}"'] & & \wp(Fm(Y)) \arrow[ll] \arrow[ul, "\Xi_Y"'] \\
G(Fm(X)) \arrow[uu] & & & \\
& \wp(Fm(X)) \arrow[uu] \arrow[ul, "\Xi_X"] & &
\end{tikzcd}$$

Sufficient for that: h is injective

$\Updownarrow$

for ~~congruence~~ Horn filter pairs

(G,h) presents an equivalent ~~alg.~~ Horn-semantics

Remaining question:

When does a filter pair have the theory lifting property?

Sufficient: Ξ is natural

$$\begin{tikzcd}
G(Fm(X\cap Y)) & & G(Fm(Y)) \arrow[ll] & \\
& \mathcal{S}(Fm(X\cap Y)) \arrow[ul, "\Xi_{X\cap Y}"'] & & \mathcal{S}(Fm(Y)) \arrow[ll] \arrow[ul, "\Xi_Y"'] \\
G(Fm(X)) \arrow[uu] & & & \\
& \mathcal{S}(Fm(X)) \arrow[ul, "\Xi_X"] \arrow[uu] & &
\end{tikzcd}$$

Sufficient for that: h is injective

Also sufficient for congruence filter pairs:

The alg. semantics is in a regular variety and given by regular equations.

Thanks!