

On the structure of involutive po-monoids as models of Multiplicative Linear Logic

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- Posets underlying residuated po-magmas and po-semigroups
- The structure of commutative ipo-monoids with a Sugihara or Boolean unital component

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Naturally ~ 0 is denoted by 1.

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Similarly $xy \leq z \iff -zx(\sim -y) \leq 0 \iff -zx \leq -y \iff y \leq \sim(-zx).$ □

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$$x \leq y \implies xz \leq yz \text{ and } zx \leq zy.$$

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We will see that ipo-monoids come in a form of **group**-indexed unions of directed (bounded) connected components $\bigcup_{g \in G} A_g$. Depending on additional axioms or assumptions on the unital component A_e , we can say a little bit more.

Examples: binary relations, pointed groups, po-groups

Any set of binary relations $A \subseteq \mathcal{P}(X^2)$ closed under composition and **complement-converse** $\sim R = -R = X^2 \setminus R^{-1}$. Similarly, **weakening relations** on a poset with composition and complement-converse.

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Examples: $(\mathbb{Z}, \leq, +, -, 0)$, $(\mathbb{R}, \leq, +, -, 0)$, $(\mathbb{Q}^+, \leq, \cdot, ^{-1}, 1)$, **all** groups (with \leq is $=$, e.g. $(\mathbb{Z}, +, -, 0)$)

Sugihara ipo-monoids (reducts of algebraic models of RM_m)

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\mathbf{S}_3 is also known as the **Gaifman-Pratt ipo-monoid**

Models from products of groups and Boolean algebras

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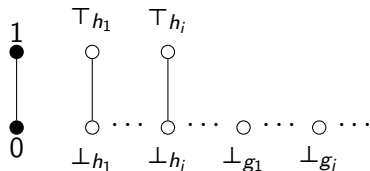
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Then \mathbf{A} is an ipo-monoid. \mathbf{A} is a **conuclear image** of $\mathbf{G} \times \mathbf{2}$ (the conucleus is the map $(g, x) \mapsto (g, \top_g \cdot \top_{g^{-1}} \cdot x)$).

Conuclei on ipo-monoids

A **conucleus** on an ipo-monoid is a **monotone interior** operator:

- $\delta(x) \leq x$
- $\delta(\delta(x)) = \delta(x)$

such that

- $\delta(x) \cdot \delta(y) \leq \delta(xy)$
- $\delta(1) = 1$
- $\delta(-(\delta(\sim\delta(x)))) \leq \delta(x)$ and $\delta(\sim(\delta(-\delta(x)))) \leq \delta(x)$

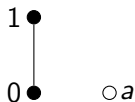
For $x \in \delta[A]$ define $\sim_{\delta}x = \delta(\sim x)$ and $-_{\delta}x = \delta(-x)$. Then

$$\mathbf{A}_{\delta} = (\delta[A], \leq, \cdot, \delta(0), \sim_{\delta}, -_{\delta})$$

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$2_{\mathbb{Z}_2, \{e\}}$ and $\mathbb{Z}_2 \times 2$

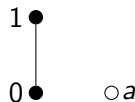
One of the simplest examples of ipo-monoids $2_{\mathbb{Z}_2, \{e\}}$ arises as a conuclear image of the product algebra $\mathbb{Z}_2 \times 2$:



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$2_{\mathbb{Z}_2, \{e\}}$ and $\mathbb{Z}_2 \times 2$

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We will later discuss inequational bases for the (po-)varieties generated by $2_{\mathbb{Z}_2, \{e\}}$ and by $\mathbb{Z}_2 \times 2$.

Partially ordered algebras

A **po-algebra** is a partially ordered set with operations that are either order-preserving **or order-reversing** in each argument.

A **variety of po-algebras** is a class of similar po-algebras defined by equations **or inequations** [Pigozzi 2004].

A **residuated partially ordered magma** or **rpo-magma** $\mathbf{A} = (A, \leq, \cdot, \backslash, /)$ is a partially-ordered set (A, \leq) with a binary operation \cdot and two **residuals** that satisfy for all $x, y, z \in A$

$$\text{(res)} \quad xy \leq z \iff x \leq z/y \iff y \leq x \backslash z$$

The operation $x \cdot y$ is usually written xy .



Residuated po-magmas are a po-variety

Residuation ensures that x/y and $y \setminus x$ are order-preserving in the numerator (x position) and **order-reversing** in the denominator.

xy is order-preserving in both arguments.

(res) is equivalent to $x \leq xy/y$, $(z/y)y \leq z$, $y \leq x \setminus xy$, $x(x \setminus z) \leq z$ hence **rpo-magmas are a variety of po-algebras**.

Although rpo-magmas are very general, (res) imposes restrictions on the posets that can occur.

E.g. could   be the poset of a rpo-magma?

For rpo-magmas, if $a, b \leq c$ then $(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a, b$.

Assume $a \leq c$.

Then $a/(a \setminus c) \leq c/(a \setminus c)$

\implies

$(c/(a \setminus c)) \setminus b \leq (a/(a \setminus c)) \setminus b$

\iff

$(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq b$

Assume $b \leq c$.

Then $a \setminus c \leq a \setminus b$

\iff

$a(a \setminus c) \leq c$

\iff

$a \leq c/(a \setminus c)$

\implies

$(c/(a \setminus c)) \setminus b \leq a \setminus b \leq a \setminus c$

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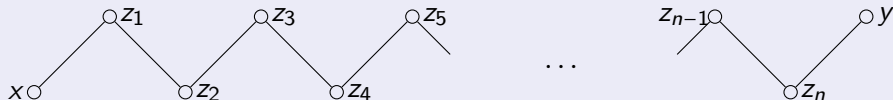
Finite rpo-magmas have bounded components

In any rpo-magma, if $d \leq a, b$ then

$$a, b \leq d / (((a \setminus d) / (a \setminus (d \setminus d))) ((d \setminus d) / (a \setminus (d \setminus d))) \setminus (b \setminus d))$$

In an rpo-magma every connected component of \leq is up-directed and down-directed, hence for **finite** rpo-magmas every connected component is **bounded**.

Two elements x, y in a poset are connected iff there exists a zigzag



We need to find an upper and a lower bound of x, y .

What posets underly rpo-magmas?

In a poset define $x \rightsquigarrow y$ if x and y are in the same connected component.

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For any po-algebra the relation \rightsquigarrow is a **congruence**.

For a rpo-magma \mathbf{A} the quotient algebra $\mathbf{A}/\rightsquigarrow$ is a **quasigroup** with \leq as equality, i.e., $xy = z \iff x = z/y \iff y = x \setminus z$.

Conversely, from any quasigroup \mathbf{Q} and a pairwise disjoint family of **bounded** posets A_q for $q \in \mathbf{Q}$, one can construct an rpo-magma with poset $\bigcup_{q \in \mathbf{Q}} A_q$.

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E.g. for a quasigroup \mathbf{Q} and $x \in A_p, y \in A_q$ define

$$x \cdot y = \perp_{pq}, \quad x \setminus y = \top_{p \setminus q}, \quad x / y = \top_{p/q}.$$

Residuated po-semigroups

A **rpo-semigroup** or **Lambek algebra** is a rpo-magma where \cdot is associative. These are algebraic models of Lambek Calculus.

*If the poset is an antichain then a rpo-semigroup is a **group**.*

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An associative quasigroup is a group: $x = y \setminus yx = y(y \setminus x)$. Hence $x = x(x \setminus x) \Rightarrow xy = x(x \setminus x)y \Rightarrow y = x \setminus xy = (x \setminus x)y \Rightarrow y/y = x \setminus x$. So $x \setminus x = e$ is constant, $xe = x$ and $ey = y$, i.e., e is an identity.

Now $(e/x)x = e \Rightarrow x(e/x)x = x \Rightarrow x(e/x) = x/x = e$, so $x^{-1} = e/x$. □

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\Rightarrow For any rpo-semigroup **A** (and therefore **for any ipo-monoid**) the quotient **A**/ \approx is a **group**.

Ipo-monoids with additional assumptions

The connected components of \mathbf{A} are denoted A_g for $g \in \mathbf{A}/\sim$, with the **unital component** $1 \in A_e$. All components are up-and-down-directed.

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The **order** of $x \in \mathbf{A}$ is the smallest positive n such that $x^n \in A_e$, or ∞ if no finite n exists. If every element has a finite order, they all divide the order of the group, so $\exists n \forall x (x^n \sim 1)$. We are interested in properties of the map $f : x \mapsto x^n$, namely if it is an order embedding and what is its image.

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In particular we look at

- Ipo-monoids where $0 \leq 1$, and A_e is idempotent
- Ipo-monoids with $x^{n+1} = x$ for some n (**n -periodic contraction**)
- Ipo-monoids where $x \leq 0$ or $1 \leq x$ for all $x \in A_e$
- weakly integral ipo-monoids ($1 \leq x \Rightarrow 1 = x$)
- Ipo-monoids where A_e is an ipo-**Sugihara monoid**, or a **Boolean algebra**.

What about cyclicity/commutativity?

- Ipo-monoids consisting of **finite chains** are **cyclic**.
- Weakly integral po-monoids with $0 \leq 1$ are **cyclic**, because their components are bounded ($0x = x0$ is the bottom element $\perp \leq x$).
- **Cyclic** ipo-monoids with a **Boolean** A_e are **commutative** if they satisfy $x^{n+1} = x$ for some $n \geq 2$. Not all cyclic ipo-monoids with a Boolean A_e are commutative.
- If (every element of \mathbf{A} has a finite order), $A_e = \{x \mid x \leq 0 \vee 1 \leq x\}$, $0 \leq 1$ and A_e is idempotent, then
 - ▶ A_e is cyclic if and only if \mathbf{A} is **balanced**
($x \cdot \sim x = -x \cdot x =: 0_x$ and $x \setminus x = x / x =: 1_x$)
 - ▶ If A_e is cyclic then A_e is a commutative subalgebra of \mathbf{A} ,
 - ▶ If A_e is cyclic and A is finite, then \mathbf{A} is cyclic.
- Ipo-monoids with a **Sugihara** $A_e = S_m$ are all **cyclic** (and therefore balanced).
- There are finite single-component non-cyclic ipo-monoids (6 elements), even chains (7 elements).

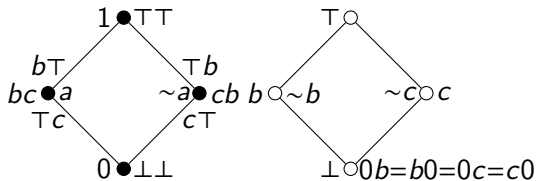


Figure: Noncommutative cyclic ipo-monoid with Boolean unital component. Note that $x^3 = x$ fails in this po-algebra since $b^3 = b \cdot \sim b \cdot b = 0b = \perp$.

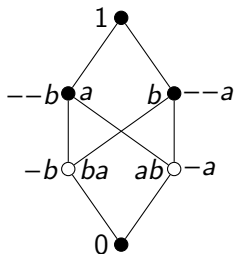


Figure: Smallest noncyclic ipo-monoid with a single connected component.

Ipo-monoids where $0 \leq 1$ (A_e is a subalgebra of \mathbf{A})

In an ipo-monoid with $0 \leq 1$, for any n , $(\sim x)^n \leq \sim x^n$ and $(-x)^n \leq -x^n$.

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Now $x \cdot x^n(\sim x)^n \cdot \sim x \leq x \cdot \sim x \leq 0$.

Hence $x^{n+1}(\sim x)^{n+1} \leq 0$ and therefore $(\sim x)^{n+1} \leq \sim x^{n+1}$. □

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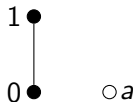
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The converse inequality $\sim x^n \leq (\sim x)^n$ is not true in general: it fails for $x = a$ and $n = 2$ in the following three-element ipo-monoid with $a = \sim a$ acting as an infectious value, and $a^2 = 0$ and $a^3 = a$:



Commutative ipo-monoids where $0 \leq 1$ and A_e idempotent

Let \mathbf{A} be a commutative ipo-monoid with a unital component that is idempotent. Then

- A_e is a locally-integral ipo-monoid. Positive elements A^+ of A_e form a join-semilattice.
- \mathbf{A} is steady: $1_x \cdot 1_y = 1_{xy}$

\mathbf{A} is commutative, and therefore cyclic and balanced. The units $1_x = x \setminus x \geq 1$ are positive, $1_x \cdot x = x$. If $x \in A_e$, then $x \leq 1_x$ and $x \setminus 1_x = 1_x$ by idempotence. By Thm. 5.1 of the paper below, **fibers $A_x = \{y \mid 1_x = 1_y\}$ of A_e are Boolean.** Proof of steadiness essentially uses commutativity. \square

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A_e can be constructed as a **Płonka sum** of its fibers A_x , indexed by the join semilattice A^+ . **Which ipo-monoids \mathbf{A} can we construct using Płonka sums?**

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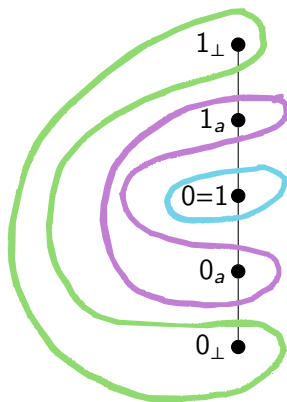
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J. Gil-Férez, P. Jipsen, M. Sugimoto: Locally integral involutive PO-semigroups, Fundamenta Informaticae, Volume 195, 1-4, 7, (2025) 1-29

Example: S_5



Commutative ipo-monoids with Boolean unital component

Let \mathbf{A} be a commutative ipo-monoid with a unital component that is a Boolean algebra \mathbf{B} and denote the group \mathbf{A}/\sim by \mathbf{G} . Then \mathbf{A} is a conuclear image of $\mathbf{G} \times \mathbf{B}$ and all components of \mathbf{A} are isomorphic to principal ideals of \mathbf{B} .

The conuclear map on $\mathbf{G} \times \mathbf{B}$ is defined by $\delta((g, x)) = (g, \top_g \cdot \top_{g^{-1}} \cdot x)$. The proof uses that $\top_g \cdot \top_{g^{-1}} \cdot \top_g = \top_g$. □

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We will use **conuclear images of $\mathbf{G} \times \mathbf{B}$** as **fibers** of **Plonka sums** to construct commutative ipo-monoids (e.g. those with Sugihara unital component).

Ipo-monoids where $0 \leq 1$, and $x \leq 0$ or $1 \leq x$ for all $x \in A_e$

Note that the assumptions hold if $A_e \simeq \mathbf{S}_k$.

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Therefore $x \cdot \sim y \leq 0$, a contradiction. □

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- All components A_g are chains.
- If all the components are finite chains, then \mathbf{A} is cyclic.
- If any component has a bottom element, all components are bounded.
- A_e is cyclic if and only if \mathbf{A} is balanced.
- If A_e is cyclic then A_e is a commutative subalgebra of \mathbf{A} ,
- If A_e is cyclic and \mathbf{A} is finite, then \mathbf{A} is cyclic.
- Assuming every element of \mathbf{A} has a finite order, $x \mapsto x^n$ is an order embedding $A_g \rightarrow A_e$, $x = x^{n+1}$, and
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- Every element is either a lower element ($f(x) \leq 0$ and $x = x \cdot \sim x \cdot x$) or an upper element ($1 \leq f(x)$ and $-x = -x \cdot x \cdot -x$).
- There is an order embedding $f : A_g \rightarrow A_e$, generated by $x \mapsto 0_x$ on lower elements (if \mathbf{A} is balanced).

Structure of ipo-monoids with $A_e \cong \mathbf{S}_k$

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We can obtain this insight without assuming finite order, if \mathbf{A} is balanced.

Structure of ipo-monoids with $A_e \cong \mathbf{S}_k$

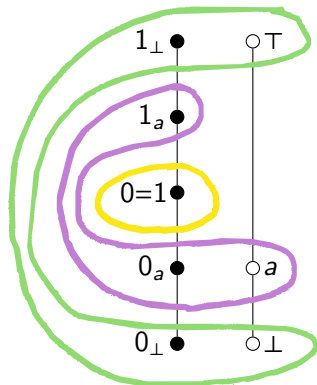


Figure: A commutative ipo-monoid with Sugihara unital component

Structure of commutative ipo-monoids with $A_e \cong \mathbf{S}_k$ - Płonka sums

Every **commutative** ipo-monoid \mathbf{A} with $A_e \cong \mathbf{S}_k$ can also be constructed using a **Płonka sum** over

- a linear join-semilattice of positive elements A^+ with the bottom element 1,
- A^+ directed system of ipo-monoids $A_x = \{y \mid 1_y = 1_x\}$ (fibers)
- $|A_x \cap A_g| \leq 2$
- fibers are ipo-monoids of the form $2_{\mathbf{G}, \mathbf{H}}$.

Conjecture: Every commutative ipo-monoid \mathbf{A} with $A_e \cong \mathbf{S}_k$ and $\mathbf{G} = \mathbf{A}/\sim$ is a conuclear image of $\mathbf{G} \times \mathbf{S}_k$.

J. Gil-Férez, P. Jipsen, M. Sugimoto: Locally integral involutive PO-semigroups, *Fundamenta Informaticae*, Volume 195, 1-4, 7, (2025) 1–29.

S. Bonzio, J. Gil-Férez, P. Jipsen, A. Přenosil, M. Sugimoto: On the structure of balanced residuated partially ordered monoids, *RAMiCS*, (2024).

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Structure of commutative ipo-monoids with $A_e \cong \mathbf{S}_k$ - Płonka sums

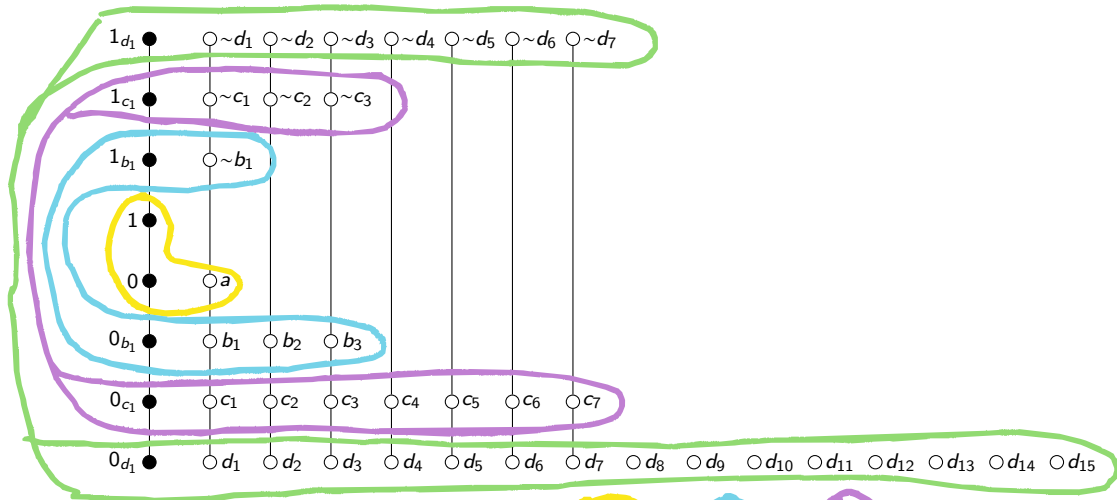


Figure: Example of a Płonka sum $2_{\mathbb{Z}_2, \{e\}} \uplus_{\varphi_1} 2_{\mathbb{Z}_2^2, H_1} \uplus_{\varphi_2} 2_{\mathbb{Z}_2^3, H_2} \uplus_{\varphi_3} 2_{\mathbb{Z}_2^4, H_3}$

Ipo-monoids with $x^{n+1} = x$ and $0 \leq 1$

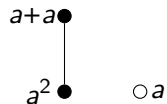
Note that **n-periodic contraction** $x^{n+1} = x$ holds for ipo-monoids of finite order, e.g. those with $A_e \cong \mathbf{S}_k$, or commutative ones with $A_e \in BA$.

- $x^{n+1} = x$ means $x \sqsubseteq x^n$ (the monoid-induced order)
- $x^{n+1} = x$ implies $x^n \rightsquigarrow 1$, and x^n is **idempotent**, and $(x^n)^n = x^n$.
- $x^{n+1} = x$ IFF $x \cdot \sim x \leq x^n \leq x \setminus x$ and $-x \cdot x \leq x^n \leq x / x$

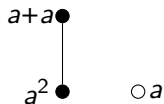
In a finite ipo-monoid A satisfying $x^{n+1} = x$ for $n = |G|$, and $0 < 1$, with $f : x \mapsto x^n$:

- Fixpoints of f coincide with the idempotents of A , and they form a cyclic balanced subalgebra of A (of C_e)
- $A' = \{x \mid x \leq 1 \text{ or } 1 \leq x\}$ is a subalgebra of $\text{Idmp}(A)$, where $\{x \mid 0 \leq x \leq 1\}$ is a Boolean algebra, and $\{x \mid x \leq 0 \text{ or } 1 \leq x\}$ is a (reduct of) Sugihara monoid.

$2_{\mathbb{Z}_2, \{e\}}$ and $\mathbb{Z}_2 \times 2$ -generated (po-)varieties



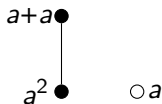
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The order is definable: $x \leq y$ **IFF** $1 = x \setminus y$.

The term $\tau(x) := \sim x^2 \cdot (x + x)$ equals **1** if $x = a$, and equals **0** otherwise.

$2_{\mathbb{Z}_2, \{e\}}$ and $\mathbb{Z}_2 \times 2$ -generated (po-)varieties



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(quasi-)inequational bases (CONJECTURE) for the varieties generated by

- $2_{\mathbb{Z}_2, \{e\}}$

- Boolean axioms instances in $(\cdot, +, \sim, 0)$ for x^2
- ipo-monoid axioms
- $x^3 = x$
- $\tau(x) \cdot \tau(y) \leq \tau(x) \cdot (\sim(xy))^2$

- $\mathbb{Z}_2 \times 2$

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- ipo-monoid axioms
- $x^3 = x$
- $\sim(x^2) = (\sim x)^2$

Thank you!



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